Isoperimetric Inequalities and topological overlapping for quotients of Affine buildings

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Abstract. We prove isoperimetric inequalities for quotients of n-dimensional Affine buildings. We use these inequalities to prove topological overlapping for the 2-dimensional skeletons of these buildings.

1 Introduction

The notion of topological overlapping was defined by Gromov in [Gro10] as:

Definition 1.1. *Let X be an n-dimensional simplicial complex. Given a map $f : X^{(0)} \to \mathbb{R}^n$ (where $X^{(0)}$ are the vertices of X), a topological extension of f is a continuous map $\widetilde{f} : X \to \mathbb{R}^n$ which coincides with f on $X^{(0)}$. A simplicial complex X is said to have c-topological overlapping (with $1 \geq c > 0$) if for every $f : X^{(0)} \to \mathbb{R}^n$ and every topological extension $\widetilde{f}$, there is a point $z \in \mathbb{R}^n$ such that*

$$|\{\sigma \in X^{(n)} : z \in \widetilde{f}(\sigma)\}| \geq c|X^{(n)}|,$$

(where $X^{(n)}$ are the n-dimensional simplices of X). In other words, this means that at least a c fraction of the images of n-simplices intersect at a single point. A family of pure n-dimensional simplicial complexes $\{X_j\}$ is called a family of topological expanders, if there is some $c > 0$ such that for every j, X_j has c-topological overlapping.

In [Gro10], Gromov gave examples of families of topological expanders, but in all the examples the degree of the vertices was unbounded. This raised the question if there are families of topological expanders with a bounded degree

on the vertices. This question received a positive answer in [KKL14], where Kaufman, Kazhdan and Lubotzky showed that the 2-skeleton of (non partite) 3-dimensional Ramanujan complexes with large enough thickness, has topological overlapping depending only on the thickness.

However, some arguments given in [KKL14] relayed heavily on the unique structure of 3-dimensional Ramanujan complexes. In this article we generalize the results of [KKL14], such that they will hold for quotients of n-dimensional (for $n > 2$) classical affine buildings of any type, assuming large enough thickness. We should remark that a lot of the ideas we use in our proofs already appear in some form in [KKL14].

As in [KKL14], we derive topological overlapping from higher isoperimetric inequalities. To state these inequalities, we shall need some additional definitions given below (this definitions are repeated and expended in subsection 3.1 below).

Let X be a pure n-dimensional simplicial complex let $-1 \leq k \leq n$. Denote by $X^{(k)}$ the set of simplices of dimension k and define a weight function

$$m : \bigcup_{k=-1}^{n} X^{(k)} \to \mathbb{R},$$

$$\forall \sigma \in \bigcup_{k=-1}^{n} X^{(k)}, m(\sigma) = \frac{(n-k)! \, |\{\eta \in X^{(n)} : \sigma \subseteq \eta\}|}{|X^{(n)}|}.$$

Denote $C^k(X, \mathbb{F}_2)$ to be the k-cochains with coefficients in $\mathbb{F}_2$, i.e., $C^k(X, \mathbb{F}_2) = \{\phi : X^{(k)} \to \mathbb{F}_2\}$ and define a norm on $C^k(X, \mathbb{F}_2)$ as

$$\|\phi\| = \sum_{\sigma \in X^{(k)}, \phi(\sigma)=1} m(\sigma).$$

Next, define the differential $d_k : C^k(X, \mathbb{F}_2) \to C^{k+1}(X, \mathbb{F}_2)$ as

$$\forall \eta \in X^{(k+1)}, d_k\phi(\eta) = \sum_{\sigma \in X^{(k)}, \sigma \subset \eta} \phi(\sigma),$$

where the addition above is in $\mathbb{F}_2$. For $k \geq 0$, define

$$B^k(X, \mathbb{F}_2) = \{d_{k-1}\phi : \phi \in C^{k-1}(X, \mathbb{F}_2)\} \subseteq C^k(X, \mathbb{F}_2).$$

A cochain $\phi \in C^k(X, \mathbb{F}_2)$ is called minimal if

$$\|\phi\| \leq \|\phi - \psi\|, \forall \psi \in B^k(X, \mathbb{F}_2).$$

Remark 1.2. *The reader should note that our definition of the norm is slightly different than the one given in [KKL14] (and in other references). This is done for technical reasons and any of our results can be easily translated to results in the norm of [KKL14]. This is explained in further detail in remark 6.2 below.*

Next, we recall the definition of local spectral expansion given by the author in [Opp14] (a reader not familiar with the definition of links can find it in subsection 2.3 below):

Definition 1.3. *A pure n-dimensional simplicial complex is said to have λ-local spectral expansion, where $\frac{n-1}{n} < \lambda \leq 1$ if both of the following hold:*

1. *All the links of X of dimension > 0 are connected.*

2. *All the 1-dimensional links have spectral gap at least λ (the 1-dimensional links are the links of the $(n-2)$-simplices).*

Our main results giving isoperimetric inequalities for n-dimensional affine buildings with large local spectral expansion. For 1-cochains an isoperimetric inequality can be deduced based on the local spectral gap alone:

Theorem 1.4. *There is a constant $\theta < 1$ such that for every pure n-dimensional simplicial complex with λ-local spectral gap with $\lambda \geq \theta$, we have that for every $\phi \in C^1(X, \mathbb{F}_2)$, if $\|\phi\| \leq 12C_1$ and ϕ is minimal, then $\|d\phi\| \geq \frac{1}{4}\|\phi\|$.*

However, in order to deduce topological overlapping, we need similar isoperimetric inequalities for 2-cochains. Currently (when writing this article), we do not know how to prove such inequalities using spectral properties alone and therefore we'll have to add an assumption regarding the coboundary expansion of the links of vertices:

Theorem 1.5. *For every $n > 2$, $\epsilon > 0$, there are constants $\frac{n-1}{n} < \Lambda_2 < 1, C_2 > 0$ such that for every pure n-simplicial complex X of dimension $n > 2$ with λ-local spectral expansion, if $\lambda \geq \Lambda_2$ and if for every $\{v\} \in X^{(0)}$*

$$\min\left\{ \frac{\|d\phi\|}{\min_{\psi \in B^1(X_{\{v\}}, \mathbb{F}_2)}\|\phi - \psi\|} : \phi \in C^1(X_{\{v\}}, \mathbb{F}_2) \setminus B^1(X_{\{v\}}, \mathbb{F}_2) \right\} \geq \epsilon,$$

(where $X_{\{v\}}$ it the link of $\{v\}$ - see further explanation below), we have for every $\phi \in C^2(X, \mathbb{F}_2)$ that

$$(\phi \text{ is minimal and } \|\phi\| \leq 24C_2) \Rightarrow \|d\phi\| \geq \frac{3\epsilon}{10}\|\phi\|.$$

Using a result stated from [KKL14] (see theorem 6.1 below), we use (a slightly more general version) of the above isoperimetric inequalities to deduce the following topological overlapping result:

Theorem 1.6. *Let $\widetilde{X}$ be an n dimensional $(n > 2)$ affine building that arises from group G with an affine BN-pair constructed over a non-archimedean local field F. Denote the thickness of $\widetilde{X}$ by t. Let Γ be a subgroup acting on $\widetilde{X}$ simplicially and cocompactly, such that for every vertex v of $\widetilde{X}$ we have that*

$$\forall g \in \Gamma, d(g.v, v) > 2.$$

Then for t large enough, there is a constant $c > 0$ that depends only on t and the type of $\widetilde{X}$ (i.e., on the Weyl group of $\widetilde{X}$), but not on Γ, such that the 2-skeleton of $\widetilde{X}/\Gamma$ has c-topological overlapping property.

Corollary 1.7. *Let $\widetilde{X}$ as in the above theorem with large thickness. Then for a sequence of groups Γ_i that act on $\widetilde{X}$ simplicially and cocompactly with the condition stated in the above theorem. Let $Y_i = \widetilde{X}/\Gamma_i$ and let X_i be the 2-skeleton of Y_i. Then $\{X_i\}$ is a family of topological expanders.*

Remark 1.8. *Explicit examples of affine buildings $\widetilde{X}$ and Γ_i's as in the corollary are the Ramanujan complexes constructed in [LSV05].*

Remark 1.9. *As noted above, it is our hope to prove the isoperimetric inequalities stated above based on spectral information only, i.e., for a general simplicial complex with large enough local spectral expansion without assuming the complex to be an affine building. The question how to do so (or if this can be done at all) is left for future study.*

Structure of this article: In section 2, we review basic definitions and results about weighted graphs, weighted complex and links. In section 3, we review different notions of high dimensional expansion considered in the article. In section 4, we prove some new technical results connecting the norm of $\mathbb{F}_2$ cochains to the norms of the localizations. In section 5, we discuss different notions of minimality for $\mathbb{F}_2$ cochains. In section 6, we discuss criteria to topological expansion and show how to drive topological expansion from (certain types of) isoperimetric inequalities. In section 7, we prove the isoperimetric inequalities stated above (which are the main results of this article). Finally, in section 8, we derive topological overlapping for the 2-skeleton of affine building. In the interest of readers not used to the weighted setting, we added an appendix covering basic results (such as the Cheeger inequality) for weighted graphs.

2 Background definitions and results

This section is aimed to provide background results that we shall need in order to prove our main theorems.

2.1 Weighted graphs and Cheeger inequalities

We shall provide the basic definitions on weighted graphs, graph Laplacians and state (without proof) some Cheeger inequalities for weighted graphs. The interested reader can find proofs and a more complete discussion regarding the definitions in the appendix.

For a graph $G = (V, E)$, a weight function is a function $m : V \cup E \to \mathbb{R}^+$ such that

$$\forall v \in V, m(v) = \sum_{e \in E, v \in e} m(e).$$

Denote by $C^0(G, \mathbb{R})$ the space:

$$C^0(G, \mathbb{R}) = \{\phi : V \to \mathbb{R}\}$$

Let $\Delta^+ : C^0(G, \mathbb{R}) \to C^0(G, \mathbb{R})$ be the graph Laplacian with respect to the weight function m which is defined as follows:

$$\Delta^+ \phi(v) = \phi(v) - \frac{1}{m(v)} \sum_{u \in V, \{u,v\} \in E} m(\{u, v\})\phi(u).$$

The Laplacian is a positive operator and if G is connected, only constant functions have the eigenvalue 0. For a connected graph G, we denote by $\lambda(G)$ the smallest positive eigenvalue. Next, we state the following Cheeger-type inequalities (the reader can find the proofs in the appendix):

Proposition 2.1. *Let $G = (V, E)$ be a connected graph. We introduce the following notations: For $\emptyset \neq U \subseteq V$ denote*

$$m(U) = \sum_{v \in U} m(v).$$

For $\emptyset \neq U_1, U_2 \subseteq V$ denote

$$m(U_1, U_2) = \sum_{u_1 \in U_1, u_2 \in U_2, \{u_1, u_2\} \in E} m(\{u_1, u_2\}).$$

Then:

1. *(Cheeger inequality) For every $\emptyset \neq U \subsetneq V$, we have that*

$$m(U, V \setminus U) \geq \lambda(G) \frac{m(U) m(V \setminus U)}{m(V)}.$$

2. *For every $\emptyset \neq U \subsetneq V$, we have that*

$$\frac{m(U)}{2} \left(1 - \lambda(G) \frac{m(V \setminus U)}{m(V)} \right) \geq m(U, U).$$

2.2 Weighted simplicial complexes

Let X be a pure n-dimensional finite simplicial complex. For $-1 \leq k \leq n$, we denote $X^{(k)}$ to be the set of all k-simplices in X ($X^{(-1)} = \{\emptyset\}$). A weight function m on X is a function:

$$m : \bigcup_{-1 \leq k \leq n} X^{(k)} \to \mathbb{R}^+,$$

such that for every $-1 \leq k \leq n - 1$ and for every $\tau \in X^{(k)}$ we have that

$$m(\tau) = \sum_{\sigma \in X^{(k+1)}} m(\sigma).$$

By its definition, it is clear the m is determined by the values in takes in $X^{(n)}$. A simplicial complex with a weight function will be called a weighted simplcial complex.

Proposition 2.2. *For every $-1 \leq k \leq n$ and every $\tau \in X^{(k)}$ we have that*

$$m(\tau) = (n - k)! \sum_{\sigma \in X^{(n)}, \tau \subseteq \sigma} m(\sigma),$$

where $\tau \subseteq \sigma$ means that τ is a face of σ.

Proof. The proof is by induction. For $k = n$ this is obvious. Assume the equality is true for $k + 1$, then for $\tau \in X^{(k)}$ we have

$$
\begin{aligned}
m(\tau) &= \sum_{\sigma \in X^{(k+1)}, \tau \subset \sigma} m(\sigma) \\
&= \sum_{\sigma \in X^{(k+1)}, \tau \subset \sigma} (n - k - 1)! \sum_{\eta \in X^{(n)}, \sigma \subset \eta} m(\eta) \\
&= (n - k)(n - k - 1)! \sum_{\eta \in X^{(n)}, \tau \subset \eta} m(\eta) \\
&= (n - k)! \sum_{\eta \in X^{(n)}, \tau \subset \eta} m(\eta).
\end{aligned}
$$

$\square$

Corollary 2.3. *For every $-1 \le k < l \le n$ and every $\tau \in X^{(k)}$ we have*

$$
m(\tau) = (l - k)! \sum_{\sigma \in X^{(l)}, \tau \subset \sigma} m(\sigma).
$$

Proof. For every $\sigma \in X^{(l)}$ we have

$$
m(\sigma) = (n - l)! \sum_{\eta \in X^{(n)}, \sigma \subseteq \eta} m(\eta).
$$

Therefore

$$
\begin{aligned}
\sum_{\sigma \in X^{(l)}, \tau \subset \sigma} m(\sigma) &= \sum_{\sigma \in X^{(l)}, \tau \subset \sigma} (n - l)! \sum_{\eta \in X^{(n)}, \sigma \subseteq \eta} m(\eta) \\
&= \frac{(n - k)!}{(l - k)! \, (n - k - (l - k))!} (n - l)! \sum_{\eta \in X^{(n)}, \tau \subseteq \eta} m(\eta) \\
&= \frac{(n - k)!}{(l - k)!} \sum_{\eta \in X^{(n)}, \tau \subseteq \eta} m(\eta) \\
&= \frac{1}{(l - k)!} m(\tau).
\end{aligned}
$$

$\square$

For $-1 \le k \le n$ and a set $\emptyset \ne U \subseteq X^{(k)}$, we denote

$$
m(U) = \sum_{\sigma \in U} m(\sigma).
$$

Proposition 2.4. *For every $-1 \le k < l \le n$,*

$$
m(X^{(k)}) = \frac{(l + 1)!}{(k + 1)!} m(X^{(l)}).
$$

Proof. By corollary 2.3, we have that

$$
\begin{aligned}
m(X^{(k)}) &= \sum_{\tau \in X^{(k)}} m(\tau) \\
&= \sum_{\tau \in X^{(k)}} (l-k)! \sum_{\sigma \in X^{(l)}, \tau \subset \sigma} m(\sigma) \\
&= \sum_{\sigma \in X^{(l)}} (l-k)! \, m(\sigma) \sum_{\tau \in X^{(k)}, \tau \subset \sigma} 1 \\
&= \sum_{\sigma \in X^{(l)}} (l-k)! \, m(\sigma) \binom{l+1}{k+1} \\
&= \frac{(l+1)!}{(k+1)!} m(X^{(l)}).
\end{aligned}
$$

$\square$

Remark 2.5. *Note that for every weighted complex X, the 1-skeleton of X is a weighted graph and all the results stated above for weighted graphs hold.*

We would like to distinguish to following weight function m_h which we call the homogeneous weight function (since it give the value 1 to each n-dimensional simplex):
$$
\forall \tau \in X^{(k)}, m_h(\tau) = (n-k)! \, |\{\eta \in X^{(n)} : \tau \subseteq \eta\}|.
$$

The next proposition shows that m_h is indeed a weight function:

Proposition 2.6. *For every $-1 \le k \le n-1$ and every $\tau \in X^{(k)}$ we have that*

$$
m_h(\tau) = \sum_{\sigma \in X^{(k+1)}, \tau \subset \sigma} m_h(\sigma).
$$

Proof. Fix $-1 \le k \le n-1$ and $\tau \in X^{(k)}$, note that for every $\eta \in X^{(n)}$ with $\tau \subset \eta$, there are exactly $n-k$ simplices $\sigma \in X^{(k+1)}$ such that $\tau \subset \sigma \subseteq \eta$. Therefore we have that

$$
\begin{aligned}
\sum_{\sigma \in X^{(k+1)}, \tau \subset \sigma} m_h(\sigma) &= (n-k-1)! \sum_{\sigma \in X^{(k+1)}, \tau \subset \sigma} |\{\eta \in X^{(n)} : \sigma \subseteq \eta\}| \\
&= (n-k)! \, |\{\eta \in X^{(n)} : \tau \subseteq \eta\}| \\
&= m_h(\tau).
\end{aligned}
$$

$\square$

It will be more convenient to normalize m_h as follows: define $\overline{m_h}$ to be the weight function:
$$
\forall \sigma \in X^{(k)}, \overline{m_h} = \frac{m_h(\sigma)}{|X^{(n)}|}.
$$

Remark 2.7. *We remark that most of our results holds for any weight function. The only place where we'll need to use the normalized homogeneous weight function is when we like to deduce topological overlap using the results in [KKL14].*

Throughout this article, X is a pure, n-dimensional weighted simplicial complex with a weight function m.

2.3   Links and spectral gaps

Let X be a pure n-dimensional finite simplicial complex. For $\{v_0, ..., v_j\} = \tau \in X^{(j)}$, denote by X_τ the *link* of τ in X, that is, the (pure) complex of dimension $n - j - 1$ consisting on simplices $\sigma = \{w_0, ..., w_k\}$ such that $\{v_0, ..., v_j\}, \{w_0, ..., w_k\}$ are disjoint as sets and $\{v_0, ..., v_j\} \cup \{w_0, ..., w_k\} \in X^{(j+k+1)}$. Note that for $\{\emptyset\} = X^{(-1)}$, $X_\emptyset = X$.

If m is a weight function on X, define m_τ to be a weight function of X_τ by taking

$$m_\tau(\sigma) = m(\tau \cup \sigma).$$

Proposition 2.8. *For $0 \leq j \leq n - 2$ and $\tau \in X^{(j)}$, the function m_τ defined above is indeed a weight function on X_τ, i.e., for every $k < n - j - 1$ and every $\sigma \in X_\tau^{(k)}$ we have that*

$$m_\tau(\sigma) = \sum_{\eta \in X_\tau^{(k+1)}} m_\tau(\eta).$$

Proof. Let $k < n - j - 1$ and $\sigma \in X_\tau^{(k)}$, then

$$\begin{aligned}
m_\tau(\sigma) &= m(\tau \cup \sigma) \\
&= \sum_{\gamma \in X^{(k+j+1)}, \tau \cup \sigma \subset \gamma} m(\gamma) \\
&= \sum_{\eta \in X^{(k)}, \tau \cup \eta \in X^{(k+j+1)}, \sigma \subset \eta} m(\tau \cup \eta) \\
&= \sum_{\eta \in X_\tau^{(k+1)}} m_\tau(\eta).
\end{aligned}$$

$\square$

By remark 2.5, the 1-skeleton of X_τ is a weighted graph with the weight function m_τ. We shall denote by $\lambda(X_\tau)$ the smallest positive eigenvalue of the (weighted) graph Laplacian on the graph.

Remark 2.9. *We remark that when one works with the homogeneous weight function m_h, then for $\tau \in X^{(n-2)}$, X_τ is a graph and $m_{h,\tau}$ assigns the weight 1 to each edge, so in this setting, the Laplacian is the usual (un-weighted) graph Laplacian.*

Next, we'll state a useful theorem from [Opp14]:

Theorem 2.10. *[Opp14][Lemma 5.1, Corollary 5.2] Let X be a pure n-dimensional weighted simplicial complex (with $n > 1$). Assume that all the links of X of dimension > 0 are connected (including X itself). For $-1 \leq k \leq n - 2$ denote*

$$\lambda_k = \min_{\tau \in X^{(k-1)}} \lambda(X_\tau).$$

1. For every $0 \leq k \leq n - 2$, we have that

$$\lambda_k \geq 2 - \frac{1}{\lambda_{k+1}}.$$

2. If $\lambda_{n-1} > \frac{n-1}{n}$, then for every $0 \leq k \leq n - 2$, we have that $\lambda_k > \frac{k}{k+1}$. Moreover, for every $N \in \mathbb{N}, N \geq n$, we have that if $\lambda_{n-1} > \frac{N-1}{N}$, then for every $0 \leq k \leq n - 2$, $\lambda_k > \frac{N-n+k}{N-n+k+1}$.

3 Different notions of expansion

Here we'll review different notions of expansion considered in this article.

3.1 $\mathbb{F}_2$-expansion

For $-1 \leq k \leq n$, define $C^k(X, \mathbb{F}_2)$ to be k-cochains of X over $\mathbb{F}_2$:

$$C^k(X, \mathbb{F}_2) = \{\phi : X^{(k)} \to \mathbb{F}_2\}.$$

For $\phi \in C^k(X, \mathbb{F}_2)$ denote by $supp(\phi)$ the support of ϕ:

$$supp(\phi) = \{\sigma \in X^{(k)} : \phi(\sigma) = 1\}.$$

Further define the differential $d_k : C^k(X, \mathbb{F}_2) \to C^{k+1}(X, \mathbb{F}_2)$ as:

$$\forall \sigma \in X^{(k+1)}, \forall \phi \in C^k(X, \mathbb{F}_2), d_k\phi(\sigma) = \sum_{\tau \in X^{(k)}, \tau \subset \sigma} \phi(\tau).$$

It is easy to check that $d_{k+1}d_k = 0$, therefore the usual cohomological definitions hold: i.e., denote

$$B^k(X, \mathbb{F}_2) = im(d_{k-1}), Z^k(X, \mathbb{F}_2) = ker(d_k), H^k(X, \mathbb{F}_2) = \frac{Z^k(X, \mathbb{F}_2)}{B^k(X, \mathbb{F}_2)}.$$

From now on, when there is no chance for confusion, we shall omit the index on the differential and just denote it by d.

Define the following norm on $C^k(X, \mathbb{F}_2)$: for every $\phi \in C^k(X, \mathbb{F}_2)$ define

$$\|\phi\| = \sum_{\tau \in X^{(k)}, \phi(\tau)=1} m(\tau).$$

Claim 3.1. *For every k as above and for every $\phi \in C^k(X, \mathbb{F}_2)$, we have that*

$$\|d\phi\| \leq \|\phi\|.$$

Proof. Fix $\phi \in C^k(X, \mathbb{F}_2)$. Note that for every $\sigma \in X^{(k+1)}$

$$(\forall \tau \in supp(\phi), \tau \not\subset \sigma) \Rightarrow d\phi(\sigma) = 0.$$

Also note that for every $\sigma \in X^{(k+1)}$, $d\phi(\sigma) \leq 1$. Therefore

$$\|d\phi\| = \sum_{\sigma \in X^{(k+1)}} m(\sigma)d\phi(\sigma) \leq \sum_{\sigma \in X^{(k+1)}} m(\sigma)\left(\sum_{\tau \in X^{(k)}, \tau \in supp(\phi), \tau \subset \sigma} 1 \right) =$$

$$\sum_{\tau \in X^{(k)}, \tau \in supp(\phi)} \sum_{\sigma \in X^{(k+1)}, \tau \subset \sigma} m(\sigma) = \sum_{\tau \in X^{(k)}, \tau \in supp(\phi)} m(\tau) = \|\phi\|.$$

$\square$

Using the above norm, we'll define the following constants:

Definition 3.2. *for $-1 \leq k \leq n-1$, and X a simplicial complex, define*

1. The k-th coboundary expansion of X:

$$\epsilon_k(X) = \min\left\{ \frac{\|d\phi\|}{\min_{\psi \in B^k(X,\mathbb{F}_2)}\|\phi - \psi\|} : \phi \in C^k(X,\mathbb{F}_2) \setminus B^k(X,\mathbb{F}_2) \right\}.$$

2. The k-th cocycle expansion of X:

$$\widetilde{\epsilon}_k(X) = \min\left\{ \frac{\|d\phi\|}{\min_{\psi \in Z^k(X,\mathbb{F}_2)}\|\phi - \psi\|} : \phi \in C^k(X,\mathbb{F}_2) \setminus Z^k(X,\mathbb{F}_2) \right\}.$$

3. The k-th cofilling constant of X:

$$\mu_k(X) = \max_{0 \neq \phi \in B^{k+1}(X,\mathbb{F}_2)} \left(\frac{1}{\|\phi\|} \min_{\psi \in C^k(X,\mathbb{F}_2), d\psi = \phi} \|\psi\| \right).$$

Remark 3.3. *Recall that $B^k(X,\mathbb{F}_2) \subseteq Z^k(X,\mathbb{F}_2)$ and therefore $\widetilde{\epsilon}_k(X) \geq \epsilon_k(X)$. Also note that*

$$\mu_k(X) = \frac{1}{\widetilde{\epsilon}_k(X)},$$

(the proof of this equality is basically unfolding the definitions of both constants and therefore it is left as an exercise to the reader).

Remark 3.4. *Note that when $k = -1$, we have that $B^{-1}(X,\mathbb{F}_2) = Z^{-1}(X,\mathbb{F}_2) = \{0\}$. Also note that the only $\phi in C^{-1}(X,\mathbb{F}_2) \setminus B^{-1}(X,\mathbb{F}_2)$ is $\phi(\emptyset) = 1$ and for that ϕ, we have that*

$$\|d\phi\| = m(X^{(0)}) = m(\emptyset) = \|\phi\|.$$

Therefore, we always have

$$\mu_{-1}(X) = \widetilde{\epsilon}_{-1}(X) = \epsilon_{-1}(X) = 1.$$

Definition 3.5. *Let $\{X_j\}$ be a family of pure n-dimensional simplicial complexes.*

1. *$\{X_j\}$ is called a family of coboundary expanders if there is a constant $\epsilon > 0$ such that*
$$\forall 0 \leq k \leq n - 1, \forall j, \epsilon_k(X_j) \geq \epsilon.$$

2. *$\{X_j\}$ is called a family of cocycle expanders if there is a constant $\epsilon > 0$ such that*
$$\forall 0 \leq k \leq n - 1, \forall j, \widetilde{\epsilon}_k(X_j) \geq \epsilon.$$

3.2 Topological expansion

Let X be an n-dimensional simplicial complex as before. Given a map $f : X^{(0)} \to \mathbb{R}^n$, a topological extension of f is a continuous map $\widetilde{f} : X \to \mathbb{R}^n$ which coincides with f on $X^{(0)}$.

Definition 3.6. *A simplicial complex X as above is said to have c-topological overlapping (with $1 \geq c > 0$) if for every $f : X^{(0)} \to \mathbb{R}^n$ and every topological extension $\widetilde{f}$, there is a point $z \in \mathbb{R}^n$ such that*

$$|\{\sigma \in X^{(n)} : z \in \widetilde{f}(\sigma)\}| \geq c|X^{(n)}|.$$

In other words, this means that at least c fraction of the images of n-simplices intersect at a single point.

A family of pure n-dimensional simplicial complexes $\{X_j\}$ is called a family of topological expanders, if there is some $c > 0$ such that for every j, X_j has c-topological overlapping.

In [Gro10], Gromov showed that any family of coboundary expanders (with respect to the homogeneous weight) is also a family of topological expanders, where the topological overlapping c is a function of ϵ defined above.

3.3 Local spectral expansion

In [Opp14], the author defined the concept of local spectral expansion:

Definition 3.7. *A pure n-dimensional simplicial complex is said to have λ-local spectral expansion, where $\frac{n-1}{n} < \lambda \leq 1$ if both of the following hold:*

1. *All the links of X of dimension > 0 are connected.*

2. *All the 1 dimensional links have spectral gap at least λ (the 1 dimensional links are the links of the $(n-2)$-simplices).*

This definition gives many results because theorem 2.10 allows one to deduce much spectral data from the spectral gap of 1-dimensional links.

4 Localization inequalities in $\mathbb{F}_2$ coefficients

Observe that for every $-1 \leq j \leq n-2$ and $\tau \in X^{(j)}$, the link X_τ is a weighted simplicial complex (with the weight m_τ) and we can define the norm of it for every $\phi \in C^k(X_\tau, \mathbb{F}_2)$. We also denote $d_{\tau,k} : C^k(X_\tau, \mathbb{F}_2) \to C^{k+1}(X_\tau, \mathbb{F}_2)$ to be the differential (as before, we shall omit the index k and denote only d_τ). Next, we'll define the concept of localization for $\phi \in C^k(X, \mathbb{F}_2)$:

Definition 4.1. *Let X be a pure n-dimensional weighted simplicial complex and let $-1 \leq j \leq n-1$. For $\tau \in X^{(j)}$, and $\phi \in C^k(X, \mathbb{F}_2)$ where $k - j - 1 \geq 0$, define the localization of ϕ at X_τ to be $\phi_\tau \in C^{k-j-1}(X_\tau, \mathbb{F}_2)$:*

$$\forall \sigma \in X_\tau^{(k-j-1)}, \phi_\tau(\sigma) = \phi(\tau \cup \sigma).$$

Remark 4.2. *When working with $C^k(X, \mathbb{R})$ (and not $C^k(X, \mathbb{F}_2)$) it is known that localization can be used to calculate norms of cochains and of differentials. Below we establish such methods when working in $C^k(X, \mathbb{F}_2)$. As far as we know, the results below are new - one can find similar results in [KKL14], proven only for Ramanujan complexes, but not for the general case. The reader should note that proposition 4.3 and lemma 4.5 below have analogous results in $C^k(X, \mathbb{R})$ that are well known (appear for instance in [BŚ97]). The reader can compare these results to the results stated in [Opp14][Lemma 4.4, Corollary 4.6].*

Proposition 4.3. *Let X be a pure n-dimensional weighted simplicial complex and let $-1 \leq j \leq n-1$. For every $k \geq j+1$ and every $\phi \in C^k(X, \mathbb{F}_2)$ we have that*

$$\binom{k+1}{j+1}\|\phi\| = \sum_{\tau \in X^{(j)}} \|\phi_\tau\|,$$

where $\|\phi_\tau\|$ is the norm of ϕ_τ in X_τ.

Proof. Let $\phi \in C^k(X, \mathbb{F}_2$ as above. Then

$$\sum_{\tau \in X^{(j)}} \|\phi_\tau\| = \sum_{\tau \in X^{(j)}} \sum_{\sigma \in X_\tau^{(k-j-1)}, \phi_\tau(\sigma)=1} m_\tau(\sigma) = \sum_{\tau \in X^{(j)}} \sum_{\tau \cup \sigma \in X^{(k)}, \phi(\tau \cup \sigma)=1} m(\tau \cup \sigma) =$$

$$\sum_{\tau \in X^{(j)}} \sum_{\gamma \in X^{(k)}, \tau \subset \gamma, \phi(\gamma)=1} m(\gamma) = \sum_{\gamma \in X^{(k)}, \phi(\gamma)=1} \sum_{\tau \in X^{(j)}, \tau \subset \gamma} m(\gamma) =$$

$$\sum_{\gamma \in X^{(k)}, \phi(\gamma)=1} \binom{k+1}{j+1} m(\gamma) = \binom{k+1}{j+1} \|\phi\|.$$

$\square$

Proposition 4.4. *Let $1 \leq k \leq n-1$ and let $\phi \in C^k(X, \mathbb{F}_2)$. Then we have that:*

$$k\|d\phi\| + \|\phi\| \geq \sum_{\{v\} \in X^{(0)}} \|d_{\{v\}}\phi_{\{v\}}\|.$$

Proof. Fix $\phi \in C^k(X, \mathbb{F}_2)$ and partition $X^{(k+1)}$ as follows: denote

$$T_0 = \{\eta \in X^{(k+1)} : \forall \sigma \in X^{(k)}, \sigma \subset \eta \Rightarrow \phi(\sigma) = 0\},$$

and for $i = 1, ..., k+2$, denote

$$T_i = \{\eta \in X^{(k+1)} : \exists! \, \sigma_1, ..., \sigma_i \in X^{(k)}, \forall j, \sigma_j \subset \eta, \phi(\sigma_j) = 1\}.$$

Then $T_0, ..., T_{k+2}$ is a disjoint partition of $X^{(k+1)}$. By the definition of the norm and the differential, we have that

$$\|\phi\| = \sum_{j=0}^{k+2} (j) m(T_j).$$

$$\|d\phi\| = \sum_{i=1}^{\lceil \frac{k+2}{2} \rceil} m(T_{2i-1}).$$

Note that for every $1 \leq i \leq k+2$ and for every $\eta \in T_i$, when choosing $v \in \eta$ there are exactly two possibilities: either $v \in \sigma_j$ for every $1 \leq j \leq i$ (where σ_j are the k-simplices as in the definition of T_i), or there is a single j_0 such that $v \notin \sigma_{j_0}$. Also note that

$$|\sigma_1 \cap ... \cap \sigma_i| = k + 2 - i.$$

12

Therefore, for every $1 \leq i \leq k+2$ and every $\eta \in T_i$ there are $k+2-i$ vertices v such that $d_{\{v\}}\phi_{\{v\}}(\eta \setminus \{v\}) = d\phi(\eta)$ and i vertices such that $d_{\{v\}}\phi_{\{v\}}(\eta \setminus \{v\}) = d\phi(\eta) + 1$ (when the addition is in $\mathbb{F}_2$). Therefore

$$\sum_{\{v\} \in X^{(0)}} \|d_{\{v\}}\phi_{\{v\}}\| = \sum_{i=1}^{\lceil \frac{k+2}{2} \rceil} (k+2-(2i-1))m(T_{2i-1}) + \sum_{i=1}^{\lfloor \frac{k+2}{2} \rfloor} (2i)m(T_{2i}).$$

We finish by:

$$\sum_{\{v\} \in X^{(0)}} \|d_{\{v\}}\phi_{\{v\}}\| - \|\phi\| =$$

$$\sum_{i=1}^{\lceil \frac{k+2}{2} \rceil} (k+2-(2i-1))m(T_{2i-1}) + \sum_{i=1}^{\lfloor \frac{k+2}{2} \rfloor} (2i)m(T_{2i}) - \sum_{j=0}^{k+2} (j)m(T_j) =$$

$$\sum_{i=1}^{\lceil \frac{k+2}{2} \rceil} (k+2-(2i-1)-(2i-1))m(T_{2i-1}) = \sum_{i=1}^{\lceil \frac{k+2}{2} \rceil} (k+4-4i)m(T_{2i-1}) \leq k\|d\phi\|.$$

$\square$

Using the above proposition we can get another result of this type:

Lemma 4.5. *Let $1 \leq k \leq n-1$ and let $\phi \in C^k(X, \mathbb{F}_2)$. Then we have that:*

$$\|d\phi\| \geq \sum_{\tau \in X^{(k-1)}} \|d_\tau \phi_\tau\| - k\|\phi\|.$$

Proof. We'll prove by induction. The case $k=1$ is proven in the above proposition. Let $k > 1$. For every $\{v\} \in X^{(0)}$ apply the induction assumption for each $\phi_{\{v\}} \in C^{k-1}(X_{\{v\}}, \mathbb{F}_2)$ to get

$$\|d_{\{v\}}\phi_{\{v\}}\| \geq \left(\sum_{\eta \in X_{\{v\}}^{(k-2)}} \|(d_{\{v\}})_\eta (\phi_{\{v\}})_\eta\| \right) - (k-1)\|\phi_{\{v\}}\| =$$

$$\left(\sum_{\tau \in X^{(k-1)}, v \in \tau} \|d_\tau \phi_\tau\| \right) - (k-1)\|\phi_{\{v\}}\|.$$

Summing on all vertices we get that

$$\sum_{\{v\} \in X^{(0)}} \|d_{\{v\}}\phi_{\{v\}}\| \geq \sum_{\{v\} \in X^{(0)}} \left(\left(\sum_{\tau \in X^{(k-1)}, v \in \tau} \|d_\tau \phi_\tau\| \right) - (k-1)\|\phi_{\{v\}}\| \right) =$$

$$\sum_{\tau \in X^{(k-1)}} k\|d_\tau \phi_\tau\| - (k+1)(k-1)\|\phi\|,$$

where in the last equality we used proposition 4.3. Next, by proposition 4.4:

$$k\|d\phi\| + \|\phi\| \geq \sum_{\{v\} \in X^{(0)}} \|d_{\{v\}}\phi_{\{v\}}\|.$$

13

Therefore

$$k\|d\phi\| + \|\phi\| \geq \sum_{\tau \in X^{(k-1)}} k\|d_\tau \phi_\tau\| - (k+1)(k-1)\|\phi\|,$$

which yields

$$\|d\phi\| \geq \sum_{\tau \in X^{(k-1)}} \|d_\tau \phi_\tau\| - k\|\phi\|,$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5 Different notions of minimality

In order to derive higher dimensional isoperimetric inequalities (see below), we must first discuss several notions of minimality of a cochain. The idea is that for $\phi \in C^k(X, \mathbb{F}_2)$ we want to measure the norm of ϕ up to $\psi \in B^k(X, \mathbb{F}_2)$. This can be done in several ways of optimality:

Definition 5.1. *Let $0 \leq k \leq n-1$ and $\phi \in C^k(X, \mathbb{F}_2)$.*

1. *ϕ is called minimal if for every $\varphi \in B^k(X, \mathbb{F}_2)$, we have that*

$$\|\phi\| \leq \|\phi - \varphi\|.$$

2. *For $1 \leq k$, ϕ is called locally minimal if for every $\{v\} \in X^{(0)}$, $\phi_{\{v\}}$ is minimal in $X_{\{v\}}$ (this definition is taken from [KKL14]), i.e., if for every $\{v\} \in X^{(0)}$ and every $\varphi \in B^{k-1}(X_{\{v\}}, \mathbb{F}_2)$ we have that*

$$\|\phi_{\{v\}}\| \leq \|\phi_{\{v\}} - \varphi\|.$$

   For $k = 0$, $\phi \in C^0(X, \mathbb{F}_2)$ is called locally minimal if it is minimal.

3. *For $\varepsilon > 0$, $k \geq 1$, $\phi \in C^k(X, \mathbb{F}_2)$ is called ε-locally minimal if for every $0 \leq j \leq k-1$ and every $\tau \in X^{(j)}$ we have that*

$$\forall \varphi \in B^{k-j-1}(X_\tau, \mathbb{F}_2), \|\phi_\tau\| \leq \|\phi_\tau - \varphi\| + \varepsilon m(\tau).$$

   For $\phi \in C^0(X, \mathbb{F}_2)$, ϕ is called ε-locally minimal if

$$\|\phi\| \leq \frac{(1+\varepsilon)m(X^{(0)})}{2}.$$

Remark 5.2. *One can show that*

ϕ is minimal $\Rightarrow \phi$ is locally minimal $\Rightarrow \phi$ is $\varepsilon-$locally minimal for every $\varepsilon > 0$.

One can also show that the reverse implications are false. We shall make no use of these facts, so the proofs are left to the reader.

Remark 5.3. *It is not hard to see that for every $\phi \in C^k(X, \mathbb{F}_2)$, if ϕ is ε-locally minimal, then for every $\tau \in X^{(k-1)}$ we have that*

$$\|\phi_\tau\| \leq \frac{(1+\varepsilon)m(\tau)}{2}.$$

The reason we consider ε-local minimality and not just local minimality (as in [KKL14]) is the following lemma:

Lemma 5.4. *For every $\phi \in C^k(X, \mathbb{F}_2)$ and every $\varepsilon > 0$ there is $\psi \in C^{k-1}(X, \mathbb{F}_2)$ such that $\phi - d\psi$ is ε-locally minimal and*

$$\|\phi\| \geq \|\phi - d\psi\| + \varepsilon\|\psi\|.$$

Proof. Fix $\varepsilon > 0$. Let $\phi \in C^k(X, \mathbb{F}_2)$. If ϕ is ε-locally minimal we are done by taking $\psi \equiv 0$. Assume that ϕ is not ε-locally minimal. If $k = 0$ and $\phi \in C^0(X, \mathbb{F}_2)$, then

$$\|\phi\| > \frac{(1 + \varepsilon)m(X^{(0)})}{2}.$$

Take $\psi \in C^{-1}(X, \mathbb{F}_2)$ to be $\psi(\emptyset) = 1$, then

$$\|\phi - d\psi\| < \frac{(1 - \varepsilon)m(X^{(0)})}{2} = \frac{(1 + \varepsilon)m(X^{(0)})}{2}$$
$$- \varepsilon m(X^{(0)}) < \|\phi\| - \varepsilon m(X^{(0)}) = \|\phi\| - \varepsilon\|\psi\|,$$

and we are done. Assume next that $k \geq 1$ and $\phi \in C^k(X, \mathbb{F}_2)$ is not ε-locally minimal. Then there is $0 \leq j \leq k - 1$ and $\tau \in X^{(j)}$, $\psi_1' \in C^{k-j-2}(X_\tau, \mathbb{F}_2)$ such that

$$\|\phi_\tau\| > \|\phi_\tau - d_\tau\psi_1'\| + \varepsilon m(\tau).$$

Define $\psi_1 \in C^{k-1}(X, \mathbb{F}_2)$ as

$$\psi_1(\eta) = \begin{cases} \psi_1'(\eta \setminus \tau) & \tau \subset \eta \\ 0 & \tau \not\subset \eta \end{cases}.$$

Note the following: for $\sigma \in X^{(k)}$, if $\tau \not\subset \sigma$ then $d\psi_1(\sigma) = 0$. Also note that for $\sigma \in X^{(k)}$, if $\tau \subset \sigma$, then

$$d\psi_1(\sigma) = \sum_{\eta \in X^{(k-1)}, \eta \subset \sigma} \psi_1(\eta)$$
$$= \sum_{\eta \in X^{(k-1)}, \eta \subset \sigma, \tau \subset \eta} \psi_1(\eta)$$
$$= \sum_{\eta \in X^{(k-1)}, \eta \subset \sigma, \tau \subset \eta} \psi_1'(\eta \setminus \tau)$$
$$= d_\tau\psi_1'(\sigma \setminus \tau)$$
$$= d_\tau\psi_1'(\sigma \setminus \tau).$$

Therefore

$$\|\phi - d\psi_1\| = \sum_{\sigma \in supp(\phi - d\psi_1), \tau \not\subset \sigma} m(\sigma) + \sum_{\sigma \in supp(\phi - d\psi_1), \tau \subset \sigma} m(\sigma) =$$
$$\sum_{\sigma \in supp(\phi), \tau \not\subset \sigma} m(\sigma) + \sum_{\tau \subset \sigma, (\sigma \setminus \tau) \in supp(\phi_\tau - d_\tau\psi_1')} m(\sigma) =$$
$$\sum_{\sigma \in supp(\phi), \tau \not\subset \sigma} m(\sigma) + \|\phi_\tau - d_\tau\psi_1'\| <$$
$$\sum_{\sigma \in supp(\phi), \tau \not\subset \sigma} m(\sigma) + \|\phi_\tau\| - \varepsilon m(\tau).$$

Note that $\|\psi_1\| = \|\psi_1'\|$ and that by corollary 2.3

$$\|\psi_1'\| \leq \sum_{\theta \in X_\tau^{(k-j-2)}} m_\tau(\theta) = \sum_{\eta \in X^{(k-1)}, \tau \subset \eta} m(\eta) = \frac{1}{(k-1-j)!} m(\tau) \leq m(\tau).$$

Combined with the previous inequality, this yields

$$\|\phi\| \geq \|\phi - d\psi_1\| + \varepsilon\|\psi_1\|.$$

Define $\phi_1 = \phi - d\psi_1$. If ϕ_1 is ε-locally minimal we are done by taking $\psi = \psi_1$. Otherwise repeat the same process to get $\psi_2 \in C^k(X, \mathbb{F}_2)$ and

$$\|\phi_1\| \geq \|\phi_1 - d\psi_2\| + \varepsilon\|\psi_2\|.$$

We continue this way until we get ϕ_l that is ε-locally minimal. Note that for every $i \leq l - 1$ we have that

$$\|\phi_i\| \geq \|\phi_{i+1}\| + \varepsilon \min\{m(\tau) : \tau \in X^{(k-1)}\},$$

and therefore this procedure ends after finitely many steps. Define $\psi = \psi_1 + ... + \psi_l$ and note that $\phi_l = \phi - d\psi$ is ε-locally minimal and that

$$\|\phi - d\psi\| \leq \|\phi_1 - d(\psi_2 + ... + \psi_l)\| - \varepsilon\|\psi_1\| \leq ... \leq \|\phi\| - \varepsilon(\|\psi_1\| + ... + \|\psi_l\|).$$

By the triangle inequality for the norm we get that

$$\|\phi\| \geq \|\phi - d\psi\| + \varepsilon\|\psi\|.$$

$\square$

6 Criteria for topological overlapping

To prove topological overlapping we'll relay on the following criterion taken from [KKL14]. In the theorem below, the norm $\|.\|$ refers to the norm with respect to the normalizes homogeneous weight $\overline{m_h}$ (also see remark below).

Theorem 6.1. *For constants $\mu > 0, \nu > 0$, there is $c = c(n, \mu, \nu) > 0$ such that if X is a finite simplicial complex (with the norm $\overline{m_h}$) satisfying:*

1. *For every $0 \leq k \leq n - 1$, we have that*

$$\mu_k(X) \leq \mu.$$

2. *For every $0 \leq k \leq n - 1$, we have that*

$$\min\{\|\phi\| : \phi \in Z^k(X, \mathbb{F}_2) \setminus B^k(X, \mathbb{F}_2)\} \geq \nu.$$

Then X has the c-topological overlapping property.

Remark 6.2. *In other reference such as [KKL14], the function that is used to define the norm is*

$$\forall \tau \in X^{(k)}, wt(\tau) = \frac{|\{\eta \in X^{(n)} : \tau \subseteq \eta\}|}{\binom{n+1}{k+1}|X^{(n)}|} = \frac{(k+1)! \overline{m_h}(\tau)}{(n+1)!},$$

16

and the norm $\|.\|_{wt}$ is defined accordingly (as we defined it using m). We did not use this function, because since it is not a proper weight function (according to our definition) and therefore using it makes some summation procedures more cumbersome. However, any inequality on the norm $\|.\|_{wt}$ can be easily transformed to an inequality about the norm $\|.\|$ defined by $\overline{m_h}$, by adding a multiplicative constant (dependent only on n and on k). Therefore we allowed ourselves to stated the theorem using our norm and not $\|.\|_{wt}$.

The idea (taken from [KKL14]) is that the conditions in the above theorem can be deduced from isoperimetric inequalities (apart from bounding μ_{n-1}).

Lemma 6.3. *Let X be a pure n-dimensional weighted simplicial complex. Assume there are constants $C_0 > 0, ..., C_{n-1} > 0$ and $\varepsilon > 0$ such that for every $0 \le k \le n-1$ and for every $0 \ne \phi \in C^k(X, \mathbb{F}_2)$ we have that*

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \le C_k m(X^{(k)}) \right) \Rightarrow \|d\phi\| > 0.$$

Denote

$$\mu = \max\{\frac{1}{\varepsilon}, \frac{1}{C_0}, \frac{2}{C_1}, ..., \frac{n}{C_{n-1}}\},$$

$$\nu = \min\{C_0, ..., C_{n-1}\}.$$

Then:

1. *For every $0 \le k \le n-2$, we have that $\mu_k(X) \le \mu$ (with $\mu_k(X)$ being the k-th cofilling constant of X defined above).*

2. *For every $0 \le k \le n-1$, we have that*

$$\min\{\|\phi\| : \phi \in Z^k(X, \mathbb{F}_2) \setminus B^k(X, \mathbb{F}_2)\} \ge \nu m(X^{(k)}).$$

Proof. We'll start by proving the second assertion. Fix $0 \le k \le n-1$ and let $\phi \in Z^k(X, \mathbb{F}_2)$. If $\|\phi\| \le C_k m(X^{(k)})$, then by lemma 5.4 there is $\psi \in C^{k-1}(X, \mathbb{F}_2)$ such that $\phi - d\psi$ is ε-locally minimal and

$$\|\phi\| \ge \|\phi - d\psi\| + \varepsilon \|\psi\|.$$

In particular, $C_k m(X^{(k)}) \ge \|\phi\| \ge \|\phi - d\psi\|$, therefore by our assumptions we have that if $\phi - d\psi \ne 0$, then

$$\|d(\phi - d\psi)\| > 0,$$

but this is a contradiction to the fact that $\phi \in Z^k(X, \mathbb{F}_2)$. Therefore $\phi = d\psi$, which mean that $\phi \in B^k(X, \mathbb{F}_2)$. This yields that for every $\phi \in Z^k(X, \mathbb{F}_2) \setminus B^k(X, \mathbb{F}_2)$, we have that $\|\phi\| > C_k m(X^{(k)})$.

Next, we'll prove the first assertion of the lemma. Fix $0 \le k \le n-2$ and let $\phi \in B^{k+1}(X, \mathbb{F}_2)$. If $\|\phi\| \ge C_{k+1} m(X^{(k+1)})$, then we have that for every $\psi \in C^k(X, \mathbb{F}_2), d\psi = \phi$, that

$$\frac{\|\psi\|}{\|\phi\|} \le \frac{m(X^{(k)})}{C_{k+1} m(X^{(k+1)})} = \frac{k+1}{C_{k+1}},$$

where the last equality is due to proposition 2.4. Assume next that $\|\phi\| \leq C_{k+1} m(X^{(k+1)})$. By lemma 5.4, there is $\psi \in C^k(X, \mathbb{F}_2)$ such that $\phi - d\psi$ is ε-locally minimal and such that

$$\|\phi\| \geq \|\phi - d\psi\| + \varepsilon \|\psi\|.$$

Therefore, $\|\phi - d\psi\| \leq C_{k+1} m(X^{(k+1)})$. Note that $\phi \in B^{k+1}(X, \mathbb{F}_2)$ and therefore $d(\phi - d\psi) = 0$. Therefore, we can deduce that $\phi - d\psi = 0$. Indeed, otherwise $\phi - d\psi$ is ε-locally minimal and $\|\phi - d\psi\| \leq C_{k+1} m(X^{(k+1)})$, which mean that $\|d(\phi - d\psi)\| > 0$, which yields a contradiction. So we have that $\phi = d\psi$ and

$$\|\phi\| \geq \|\phi - d\psi\| + \varepsilon \|\psi\| = \varepsilon \|\psi\|.$$

Therefore

$$\frac{\|\psi\|}{\|\phi\|} \leq \frac{1}{\varepsilon},$$

and we are done. $\qquad\square$

Using the above lemma combined with the criterion for topological overlap stated above, we can deduce the following:

Lemma 6.4. *Let $1 \leq l \leq n-1, M \geq 1, C_0 > 0, ..., C_l > 0, \varepsilon > 0$. There is a constant $c = c(M, C_0, ..., C_l, \varepsilon)$ such that for every simplicial complex X of dimension $n > 1$, if*

1. For every $0 \leq k \leq l$ and for every $0 \neq \phi \in C^k(X, \mathbb{F}_2)$ we have that

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \leq C_k \overline{m_h}(X^{(k)}) \right) \Rightarrow \|d\phi\| > 0.$$

2. We have that
$$\frac{\sup_{\sigma \in X^{(l)}} |\{\eta \in X^{(n)} : \sigma \subset \eta\}|}{\inf_{\sigma \in X^{(l)}} |\{\eta \in X^{(n)} : \sigma \subset \eta\}|} \leq M.$$

Then the l-skeleton of X has c-topological overlapping property.

Proof. In lemma 6.3, we showed that there are $\mu = \mu(C_0, ..., C_l, \varepsilon), \nu = \nu((C_0, ..., C_l)$ such that

1. For every $0 \leq k \leq l-1$, we have that $\mu_k(X) \leq \mu$ (with $\mu_k(X)$ being the k-th cofilling constant of X defined above).

2. For every $0 \leq k \leq l-1$, we have that

$$\min\{\|\phi\| : \phi \in Z^k(X, \mathbb{F}_2) \setminus B^k(X, \mathbb{F}_2)\} \geq \nu \overline{m_h}(X^{(k)}) = \nu \frac{(n+1)!}{(k+1)!}.$$

However, one should note that we cannot apply theorem 6.1 to the l-skeleton yet, since we have to deal with the following issue: all the inequalities states above refer to the norm $\overline{m_h}$ on X that is based on the n-dimensional simplices. To apply theorem 6.1 on the l-skeleton, we shall need similar inequalities when all the norms are computed with respect to the norm $\overline{m_{h,l}}$ defined using the l simplices as:

$$\forall 0 \leq k \leq l, \forall \tau \in X^{(k)}, \overline{m_{h,l}}(\tau) = \frac{(l-k)! |\{\eta \in X^{(l)} : \tau \subseteq \eta\}|}{|X^{(l)}|}.$$

Therefore, we'll need to compare the norm calculated by $\overline{m_{h,l}}$ to the norm calculated by $\overline{m_h}$. Denote the norm with respect to $\overline{m_{h,l}}$ as $\|.\|_l$. Also denote

$$M_1 = \inf_{\sigma \in X^{(l)}} |\{\eta \in X^{(n)} : \sigma \subset \eta\}|,$$

$$M_2 = \sup_{\sigma \in X^{(l)}} |\{\eta \in X^{(n)} : \sigma \subset \eta\}|.$$

Then we have that for every $\sigma \in X^{(l)}$ that

$$M_1 \leq |\{\eta \in X^{(n)} : \sigma \subset \eta\}| \leq M_2,$$

and

$$\frac{M_1}{\binom{n+1}{l+1}} |X^{(l)}| \leq |X^{(n)}| \leq \frac{M_2}{\binom{n+1}{l+1}} |X^{(l)}|.$$

Therefore we have for every $\sigma \in X^{(l)}$ that

$$\frac{(n-l)! \binom{n+1}{l+1}}{M} \overline{m_{h,l}}(\sigma) = \frac{(n-l)! M_1}{\frac{M_2}{\binom{n+1}{l+1}} |X^{(l)}|} \leq \frac{(n-l)! |\{\eta \in X^{(n)} : \sigma \subset \eta\}|}{|X^{(n)}|} = \overline{m_h}(\sigma).$$

Similarly for every $\sigma \in X^{(l)}$

$$\overline{m_h}(\sigma) \leq \frac{(n-l)! \binom{n+1}{l+1} M_2}{M_1 |X^{(l)}|} = (n-l)! \binom{n+1}{l+1} M \overline{m_{h,l}}(\sigma).$$

Therefore, by the definition of the weight function, we have that for every $0 \leq k \leq l$ and every $\tau \in X^{(k)}$ the following

$$\frac{(n-l)! \binom{n+1}{l+1}}{M} \overline{m_{h,l}}(\tau) \leq \overline{m_h}(\tau) \leq (n-l)! \binom{n+1}{l+1} M \overline{m_{h,l}}(\tau).$$

This in turn yields that for every $0 \leq k \leq l$ and every $\phi \in C^k(X, \mathbb{F}_2)$ we have that

$$\frac{(n-l)! \binom{n+1}{l+1}}{M} \|\phi\|_l \leq \|\phi\| \leq (n-l)! \binom{n+1}{l+1} M \|\phi\|_l.$$

Therefore any inequality stated in the usual norm $\|.\|$ of X can be transformed to an inequality in the "l-skeleton norm" (the constants may change as M changes). Explicitly, let X' be the l-skeleton of X, with the norm $\|.\|_l$, then with μ, ν as above we have that

1. For every $0 \leq k \leq l-1$, we have that $\mu_k(X') \leq M^2 \mu$.

2. For every $0 \leq k \leq l-1$, we have that

$$\min\{\|\phi\|_l : \phi \in Z^k(X', \mathbb{F}_2) \setminus B^k(X', \mathbb{F}_2)\} \geq \frac{1}{M} \frac{(l+1)!}{(k+1)!} \nu.$$

Therefore we are done by applying theorem 6.1 to X'. $\qquad\square$

7 Isoperimetric inequalities

In this section we shall prove the main result of this paper. The main idea of this result and its proof are taken from [KKL14]. Following [KKL14], we define the notion of thick and thin:

Definition 7.1. *Let $0 < \delta < 1, 0 < r \leq 1$ and X be a pure n-dimensional weighted simplicial complex. Define the following:*

1. *$\phi \in C^0(X, \mathbb{F}_2)$ will be called δ-thin if*

$$\|\phi\| \leq \delta m(X^{(0)}).$$

   Otherwise, we shall call ϕ δ-thick.

2. *For $k > 0$ and $\phi \in C^k(X, \mathbb{F}_2)$, we shall call $\tau \in X^{(k-1)}$, δ-thin, is $\phi_\tau \in C^0(X_\tau, \mathbb{F}_2)$ if δ-thin, i.e., if*

$$\|\phi_\tau\| \leq \delta m(\tau).$$

   Otherwise, we shall call τ δ-thick. Denote

$$A_\delta = \{\tau \in X^{(k-1)} : \tau \text{ is } \delta\text{-thin}\}.$$

3. *For $k > 0$ and $\phi \in C^k(X, \mathbb{F}_2)$, we shall call ϕ (r, δ)-thin, if*

$$\sum_{\tau \in A_\delta} \|\phi_\tau\| \geq r(k+1)\|\phi\|.$$

   Otherwise, ϕ will be called (r, δ)-thick.

Lemma 7.2. *Let X be a pure n-dimensional weighted simplicial. Let $0 < \varepsilon < 1, 0 < \delta < \frac{1}{2}, 0 < r \leq 1$. Denote*

$$\lambda_0 = \lambda(X),$$

$$k \geq 1, \lambda_k = \min_{\tau \in X^{(k-1)}} \lambda(X_\tau),$$

(see 2.3 to recall the definition of $\lambda(X_\tau)$ and some facts about it).

1. *If $\phi \in C^0(X, \mathbb{F}_2)$ is δ-thin, then*

$$\|d\phi\| \geq \lambda_0(1-\delta)\|\phi\|.$$

2. *For $k > 0$, if $\phi \in C^k(X, \mathbb{F}_2)$ is ε-locally minimal and (r, δ)-thin, then*

$$\|d\phi\| \geq \left(\left(\frac{r+1}{2} - \delta - \frac{\varepsilon}{2}\right)\lambda_k(k+1) - k\right)\|\phi\|.$$

Proof. 1. Let $\phi \in C^0(X, \mathbb{F}_2)$ that is δ-thin. Note that by definition

$$\|d\phi\| = m(supp(\phi), X^{(0)} \setminus supp(\phi)),$$

20

and
$$\|\phi\| = m(supp(\phi)).$$

Therefore, the assumption that ϕ is δ-thin, yields that
$$m(supp(\phi)) \leq \delta m(X^{(0)}),$$

and equivalently that
$$m(X^{(0)} \setminus supp(\phi)) \geq (1-\delta)m(X^{(0)}).$$

By proposition 2.1, we have that
$$m(supp(\phi), X^{(0)} \setminus supp(\phi)) \geq \lambda_0 \frac{m(supp(\phi))m(X^{(0)} \setminus supp(\phi))}{m(X^{(0)})} \geq \lambda_0(1-\delta)\|\phi\|,$$

and we are done.

2. For $k > 0$, let $\phi \in C^k(X, \mathbb{F}_2)$ be ε-locally minimal and (r, δ)-thin. Note that for every $\tau \in X^{(k-1)}$, X_τ is a weighted graph with
$$\lambda(X_\tau) \geq \lambda_k.$$

By the assumption that ϕ is ε-locally minimal, we have for every $\tau \in X^{(k-1)}$ that ϕ_τ is $\frac{1+\varepsilon}{2}$-thin (by remark 5.3). Therefore by result above for the case $k = 0$, we have for every $\tau \in X^{(k-1)}$ that
$$\|d_\tau \phi_\tau\| \geq \lambda_k \frac{1-\varepsilon}{2}\|\phi_\tau\|.$$

For $\tau \in A_\delta$, we have by the result for $k = 0$ that
$$\|d_\tau \phi_\tau\| \geq \lambda_k(1-\delta)\|\phi_\tau\|.$$

Therefore
$$\sum_{\tau \in X^{(k-1)}} \|d_\tau \phi_\tau\| = \sum_{\tau \in A_\delta} \|d_\tau \phi_\tau\| + \sum_{\tau \in X^{(k-1)} \setminus A_\delta} \|d_\tau \phi_\tau\| \geq$$
$$\lambda_k \left((1-\delta)\sum_{\tau \in A_\delta} \|\phi_\tau\| + \frac{1-\varepsilon}{2}\sum_{\tau \in X^{(k-1)} \setminus A_\delta} \|\phi_\tau\| \right) =$$
$$\lambda_k \left(\frac{1}{2}\sum_{\tau \in X^{(k-1)}} \|\phi_\tau\| + (\frac{1}{2}-\delta)\sum_{\tau \in A_\delta} \|\phi_\tau\| - \frac{\varepsilon}{2}\sum_{\tau \in X^{(k-1)} \setminus A_\delta} \|\phi_\tau\| \right) =$$
$$\lambda_k \left(\frac{k+1}{2}\|\phi\| + (\frac{1}{2}-\delta)\sum_{\tau \in A_\delta} \|\phi_\tau\| - \frac{\varepsilon}{2}\sum_{\tau \in X^{(k-1)} \setminus A_\delta} \|\phi_\tau\| \right).$$

Next, by the assumption that ϕ is (r, δ)-thin we get by the above proposition that
$$(\frac{1}{2}-\delta)\sum_{\tau \in A_\delta} \|\phi_\tau\| - \frac{\varepsilon}{2}\sum_{\tau \in X^{(k-1)} \setminus A_\delta} \|\phi_\tau\| \geq$$
$$\left(r(k+1)(\frac{1}{2}-\delta) - (1-r)(k+1)\frac{\varepsilon}{2} \right)\|\phi\|.$$

Therefore we showed that

$$\sum_{\tau \in X^{(k-1)}} \|d_\tau \phi_\tau\| \geq \left(\frac{1}{2} + r(\frac{1}{2} - \delta) - (1-r)\frac{\varepsilon}{2} \right) \lambda_k(k+1)\|\phi\| =$$

$$\left(\frac{r+1}{2} - r\delta - (1-r)\frac{\varepsilon}{2} \right) \lambda_k(k+1)\|\phi\|.$$

By lemma 4.5, we have that

$$\|d\phi\| \geq \sum_{\tau \in X^{(k-1)}} \|d_\tau \phi_\tau\| - k\|\phi\|.$$

Therefore

$$\|d\phi\| \geq \left(\left(\frac{r+1}{2} - r\delta - (1-r)\frac{\varepsilon}{2} \right) \lambda_k(k+1) - k \right) \|\phi\|.$$

From the fact that $0 < r \leq 1$, this yields that

$$\|d\phi\| \geq \left(\left(\frac{r+1}{2} - \delta - \frac{\varepsilon}{2} \right) \lambda_k(k+1) - k \right) \|\phi\|.$$

$\square$

A simple corollary of the above lemma is the following isoperimetric inequality for the case $k = 0$:

Corollary 7.3. *For X that is a pure n-dimensional weighted simplicial complex with $n \geq 1$, denote*

$$\lambda_0 = \lambda(X).$$

For every X as above, if $\lambda_0 > 0$ then for every $1 > \varepsilon > 0$, we have for every $0 \neq \phi \in C^0(X, \mathbb{F}_2)$ that

$$(\phi \text{ is } \varepsilon\text{-locally minimal}) \Rightarrow \|d\phi\| \geq \lambda_0 \frac{1-\varepsilon}{2}\|\phi\| > 0.$$

Proof. Fix $1 > \varepsilon > 0$. By definition we have that for every $\phi \in C^0(X, \mathbb{F}_2)$, if ϕ is ε-locally minimal, then

$$\|\phi\| \leq \frac{1+\varepsilon}{2} m(X^{(0)}).$$

By the above lemma, we have that

$$\|d\phi\| \geq \lambda_0(1 - \frac{\varepsilon}{2})\|\phi\| > 0,$$

and we are done. $\square$

Next, we shall prove the following:

Lemma 7.4. *For X a pure n dimensional weighted simplicial complex of dimension $n > 1$ denote*

$$\lambda_0 = \lambda(X).$$

For every $\delta > 0, \varepsilon_1 > 0$, there are constants $0 < C_1' = C_1'(\delta, \varepsilon_1) < 1, \theta_1' = \theta_1'(\delta, \varepsilon_1) < 1$, such that if X is as above with

$$\lambda_0 \geq \theta_1',$$

then for every $\phi \in C^1(X, \mathbb{F}_2)$, if $\|\phi\| \leq C_1' m(X^{(1)})$ then:

1.

$$2\varepsilon_1 \|\phi\| \geq \sum_{\{u,v\} \in X^{(1)}, \{u\} \notin A_\delta, \{v\} \notin A_\delta} m(\{u,v\}).$$

2. ϕ *is $(\frac{1}{2} - \varepsilon_1, \delta)$-thin.*

Proof. Fix $\delta > 0, \varepsilon_1 > 0$. Choose

$$C_1' = \delta^2 \varepsilon_1,$$

$$\theta_1' = 1 - \delta \varepsilon_1.$$

Assume that $\lambda_0 \geq \theta_1'$ and let $\phi \in C^1(X, \mathbb{F}_2)$ such that $\|\phi\| \leq C_1' m(X^{(1)})$. In the case $k = 1$, $A_\delta \subset X^{(0)}$. We'll denote

$$R^1 = A_\delta, S^1 = X^{(0)} \setminus S^1.$$

(these notations are in order to make the proof of this lemma similar to the proof of theorem 7.8 below).

Note that for every $\{v\} \in S^1$ we have that

$$\|\phi_{\{v\}}\| \geq \delta m(v).$$

By proposition 4.3, we have that

$$2\|\phi\| \geq \sum_{\{v\} \in S^1} \|\phi_{\{v\}}\| \geq \sum_{\{v\} \in S^1} \delta m(v) = \delta m(S^1).$$

Therefore

$$\frac{2}{\delta} \|\phi\| \geq m(S^1), \tag{1}$$

and

$$2C_1' m(X^{(1)}) \geq \delta m(S^1).$$

By the choice of C_1' and since $m(X^{(0)}) = 2m(X^{(1)})$ we get that

$$\delta \varepsilon_1 \geq \frac{m(S^1)}{m(X^{(0)})}.$$

Equivalently,

$$1 - \delta \varepsilon_1 \leq \frac{m(R^1)}{m(X^{(0)})}. \tag{2}$$

23

Recall that by proposition 2.1, we have that

$$\frac{m(S^1)}{2}\left(1 - \lambda_0 \frac{m(R^1)}{m(X^{(0)})}\right) \geq m(S^1, S^1).$$

Combining the above inequality with (1), (2) and the choice of θ_1' we get that

$$\frac{1}{\delta}\|\phi\|\left(1 - (1 - \delta\varepsilon_1)^2\right) \geq m(S^1, S^1).$$

This yields that

$$2\varepsilon_1\|\phi\| \geq m(S^1, S^1). \qquad (3)$$

Next, for $i = 0, 1, 2$ denote the following sets

$$K_i^1 = \{\sigma \in X^{(1)} : |\{\{v\} \in S^1 : \{v\} \subset \sigma\}| = i\}.$$

Note that $X^{(1)} = K_0^1 \cup K_1^1 \cup K_2^1$ and all the above sets are disjoint. With this notation we reinterpret (3) as

$$2\varepsilon_1\|\phi\| \geq m(K_2^1) \geq m(K_2^1 \cap supp(\phi)).$$

This proves the first assertion in the lemma. The above inequality yields that

$$(1 - 2\varepsilon_1)\|\phi\| \leq m(K_0^1 \cap supp(\phi)) + m(K_1^1 \cap supp(\phi)).$$

Also note that

$$\sum_{\{v\} \in R^1} \|\phi_{\{v\}}\| = 2m(K_0^1 \cap supp(\phi)) + m(K_1^1 \cap supp(\phi)).$$

Therefore

$$(1 - 2\varepsilon_1)\|\phi\| \leq 2 \sum_{\{v\} \in R^1} \|\phi_{\{v\}}\|,$$

which yields

$$2(\frac{1}{2} - \varepsilon_1)\|\phi\| \leq \sum_{\{v\} \in R^1} \|\phi_{\{v\}}\|,$$

as needed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Combining the two lemmas above we get the following isoperimetric inequality :

Theorem 7.5. *For X that is a pure n-dimensional weighted simplicial complex with $n > 1$, denote*

$$\lambda_0 = \lambda(X),$$

$$\lambda_1 = \min_{\{v\} \in X^{(0)}} \lambda(X_{\{v\}}).$$

There are constants $\varepsilon > 0, \theta_1 = \theta_1 < 1, C_1 = C_1 > 0$ such that for every X as above we have that if $\min\{\lambda_0, \lambda_1\} \geq \theta_1$, then for every $0 \neq \phi \in C^1(X, \mathbb{F}_2)$

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \leq C_1 m(X^{(1)})\right) \Rightarrow \|d\phi\| \geq \frac{1}{4}\|\phi\| > 0.$$

Proof. Take

$$\delta = \varepsilon = \frac{1}{16}, \varepsilon_1 = \frac{1}{32}.$$

With the above choice let $\theta_1' = \theta_1'(\delta, \varepsilon_1), C_1' = C_1'(\delta, \varepsilon_1)$ as in lemma 7.4. Next take $C_1 = C_1'(\delta, \varepsilon_1)$,

$$\theta_1 = \max\left\{\theta_1', \frac{\frac{1}{4}}{\frac{1}{2} - \frac{7}{32}}\right\}.$$

Let X with $\min\{\lambda_0, \lambda_1\} \geq \theta_1$ and let $\phi \in C^1(X, \mathbb{F}_2)$ such that ϕ is ε-locally minimal and $\|\phi\| \leq C_1 m(X^{(1)})$. By lemma 7.4, we have that ϕ is $(\frac{1}{2} - \frac{1}{32}, \delta)$-thin. By lemma 7.2 for $k = 1$, we get that

$$\|d\phi\| \geq \left(\lambda_1(1 + \frac{1}{2} - \frac{1}{32} - 2\delta - \varepsilon) - 1\right)\|\phi\|.$$

By the choice of $\delta, \varepsilon, \varepsilon_1$ this yields

$$\|d\phi\| \geq \left(\lambda_1(1 + \frac{1}{2} - \frac{1}{32} - \frac{1}{8} - \frac{1}{16} - 1\right)\|\phi\|.$$

After simplifying, we get that

$$\|d\phi\| \geq \left(\lambda_1(\frac{1}{2} - \frac{7}{32})\right)\|\phi\|.$$

To finish, recall that by the choice of θ_1, we have that $\lambda_1 \geq \frac{\frac{1}{4}}{\frac{1}{2} - \frac{7}{32}}$ and therefore

$$\|d\phi\| \geq \frac{1}{4}\|\phi\|.$$

$\square$

Remark 7.6. *Our choice of $\frac{1}{4}$ in the above theorem is arbitrary: for every $\varepsilon_1' > 0$, one can find $C_1, \theta_1, \varepsilon$, such that the formulation of the above theorem reads*

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \leq C_1 m(X^{(1)})\right) \Rightarrow \|d\phi\| \geq \left(\frac{1}{2} - \varepsilon_1'\right)\|\phi\|.$$

Remark 7.7. *The above theorem provides an isoperimetric inequality for the case $k = 1$ (i.e., for $\phi \in C^1(X, \mathbb{F}_2)$ under certain conditions). Note that this result does not depend on anything other than the spectral properties of the simplicial complex and can be deduced from large enough local spectral expansion. At this point, we do not know how to prove isoperimetric inequality in the $k = 2$ cases strictly from spectral gap considerations (it is our hope to do so in the future). Below, we will show the isoperimetric inequality for the case $k = 2$ under further assumptions.*

Theorem 7.8. *For X that is a pure n-dimensional weighted simplicial complex with $n > 2$, denote*

$$\lambda_0 = \lambda(X),$$

$$\lambda_1 = \min_{\{v\} \in X^{(0)}} \lambda(X_{\{v\}}),$$

$$\lambda_2 = \min_{\tau \in X^{(1)}} \lambda(X_\tau).$$

For every $1 \geq \varepsilon_2 > 0, \delta > 0$ there are constants $\theta'_2 = \theta'_2(\varepsilon_2, \delta) < 1, C'_1 = C'_1(\varepsilon_2, \delta) > 0, C'_2 = C'_2(\varepsilon_2, \delta) > 0$ such that for every X as above and $\phi \in C^2(X, \mathbb{F}_2)$ we have that if:

1. $\min\{\lambda_0, \lambda_1, \lambda_2\} \geq \theta'_2$.

2. $\|\phi\| \leq C'_2 m(X^{(2)})$.

Then one of the following holds:

1. There is a set $S^2 \subset X^{(0)}$ such that

$$\forall \{v\} \in S^2, \|\phi_{\{v\}}\| \geq C'_1 m(X^{(1)}_{\{v\}}),$$

$$\sum_{\{v\} \in S^2} \|\phi_{\{v\}}\| \geq \frac{9}{20}\|\phi\|,$$

and

$$\|d\phi\| \geq \sum_{\{v\} \in S^2} \|d_{\{v\}}\phi_{\{v\}}\| - \frac{11\varepsilon_2}{9} \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|.$$

2. ϕ is $(\frac{1}{3} + \frac{\varepsilon_2}{15}, \delta)$-thin.

Proof. Fix $1 \geq \varepsilon_2 > 0, \delta > 0$. Denote $\varepsilon_1 = \frac{\varepsilon_2}{60}$ and let $C'_1 = C'_1(\varepsilon_1, \delta), \theta'_1 = \theta'_1(\varepsilon_1, \delta)$ be the constants from lemma 7.4 above. Choose

$$\theta'_2 = \max\left\{\theta'_1, 1 - \frac{C'_1 \varepsilon_2}{60}\right\},$$

$$C'_2 = \frac{(C'_1)^2 \varepsilon_2}{60},$$

Let X as above with $\min\{\lambda_0, \lambda_1, \lambda_2\} \geq \theta'_2$. Let $\phi \in C^2(X, \mathbb{F}_2)$ such that $\|\phi\| \leq C'_2 m(X^{(2)})$. Partition the vertices of $X^{(0)}$ as follows:

$$R^2 = \{\{v\} \in X^{(0)} : \|\phi_{\{v\}}\| \leq C'_1 m(X^{(1)}_{\{v\}})\},$$

$$S^2 = X^{(0)} \setminus R^2.$$

Note that for every $v \in X^{(0)}$, we have that

$$\lambda_0(X_{\{v\}}) = \lambda(X_{\{v\}}) \geq \lambda_1(X).$$

Also note that

$$3C'_2 m(X^{(2)}) \geq 3\|\phi\| \geq \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\| \geq \sum_{\{v\} \in S^2} C'_1 m(X^{(1)}_{\{v\}})$$
$$= \sum_{\{v\} \in S^2} C'_1 \frac{1}{2} m(v)$$
$$= \frac{1}{2} C'_1 m(S^2).$$

Therefore
$$\frac{6}{C_1'}\|\phi\| \geq m(S^2),$$

$$\frac{C_1'\varepsilon_2}{60} = \frac{C_2'}{C_1'} \geq \frac{m(S^2)}{6m(X^{(2)})} = \frac{m(S)}{m(X^{(0)})}.$$

The last inequality yields that
$$\frac{m(R^2)}{m(X^{(0)})} \geq 1 - \frac{C_1'\varepsilon_2}{60}.$$

As in the proof of lemma 7.4, we imply proposition 2.1 and get that
$$\frac{m(S^2)}{2}\left(1 - \frac{m(R^2)}{m(X^{(0)})}\right) \geq m(S^2, S^2).$$

Therefore by the inequalities above and the choice of θ_2', we get that
$$\frac{3}{C_1'}\|\phi\|\left(1 - \left(1 - \frac{C_1'\varepsilon_2}{60}\right)^2\right) \geq m(S^2, S^2).$$

Therefore
$$\frac{\varepsilon_2}{10}\|\phi\| \geq m(S^2, S^2).$$

For $i = 0, 1, 2, 3$ denote
$$K_i^2 = \{\sigma \in X^{(2)} : |\{\{v\} \in S^2 : \{v\} \subset \sigma\}| = i\}.$$

$\{K_i^2\}_{i=0}^3$ is a partition of $X^{(2)}$ and therefore
$$\|\phi\| = \sum_{i=0}^3 m(K_i^2 \cap supp(\phi)).$$

Note that
$$m(K_2^2) + 3m(K_3^2) = \sum_{\sigma \in K_2^2} m(\sigma) + \sum_{\sigma \in K_3^2} 3m(\sigma)$$
$$= \sum_{\sigma \in K_2^2}\left(\sum_{\{u,v\}\in X^{(1)},\{u,v\}\subset\sigma,\{u\}\in S^2,\{v\}\in S^2} m(\sigma)\right)$$
$$+ \sum_{\sigma \in K_3^2}\left(\sum_{\{u,v\}\in X^{(1)},\{u,v\}\subset\sigma,\{u\}\in S^2,\{v\}\in S^2} m(\sigma)\right)$$
$$= \sum_{\{u,v\}\in X^{(1)},\{u\}\in S^2,\{v\}\in S^2}\left(\sum_{\sigma\in X^{(3)},\{u,v\}\subset\sigma} m(\sigma)\right)$$
$$= \sum_{\{u,v\}\in X^{(1)},\{u\}\in S^2,\{v\}\in S^2} m(\tau)$$
$$= m(S^2, S^2).$$

Therefore
$$\frac{\varepsilon_2}{10}\|\phi\| \geq m(K_2^2) + 3m(K_3^2) \geq m(K_2^2) + m(K_3^2). \tag{4}$$

This yields that

$$(1 - \frac{\varepsilon_2}{10})\|\phi\| \le m(K_0^2 \cap supp(\phi)) + m(K_1^2 \cap supp(\phi)). \tag{5}$$

Denote

$$\alpha = \frac{m(K_0^2 \cap supp(\phi))}{m(K_1^2 \cap supp(\phi))},$$

(we'll also deal with the case where $m(K_1^2 \cap supp(\phi)) = 0$). Next, we'll prove two inequalities dependant on the values of α, which much the two options stated in the theorem.

Inequality 1 : Notice that

$$\sum_{\{v\} \in S^2} \|\phi_{\{v\}}\| = m(K_1^2 \cap supp(\phi)) + 2m(K_2^2 \cap supp(\phi))$$

$$+ 3m(K_3^2 \cap supp(\phi)) \ge m(K_1^2 \cap supp(\phi)).$$

Therefore, by (5), we have that

$$(1 - \frac{\varepsilon_2}{10})\|\phi\| \le (1+\alpha)m(K_1^2 \cap supp(\phi)) \le (1+\alpha)\sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|.$$

Using the fact that $\varepsilon_2 \le 1$, this yields

$$\|\phi\| \le \frac{10(1+\alpha)}{9} \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|. \tag{6}$$

Next, we'll analyse the connection between $d\phi$ and $d_{\{v\}}\phi_{\{v\}}$ for $\{v\} \in S^2$. For $i = 0, 1, 2, 3, 4$, denote

$$K_i^3 = \{\sigma \in X^{(3)} : |\{\{v\} \in S^2 : \{v\} \subset \sigma\}| = i\}.$$

As before, we have that

$$m(K_2^3) + 3m(K_3^3) + 6m(K_4^3) = \sum_{\eta \in K_2^3} m(\eta) + \sum_{\eta \in K_3^2} 3m(\eta) + \sum_{\eta \in K_4^2} 3m(\eta)$$

$$= \sum_{\sigma \in K_3^2} \left(\sum_{\{u,v\} \in X^{(1)}, \{u,v\} \subset \eta, \{u\} \in S^2, \{v\} \in S^2} m(\eta) \right)$$

$$+ \sum_{\sigma \in K_3^2} \left(\sum_{\{u,v\} \in X^{(1)}, \{u,v\} \subset \eta, \{u\} \in S^2, \{v\} \in S^2} m(\eta) \right)$$

$$+ \sum_{\sigma \in K_4^2} \left(\sum_{\{u,v\} \in X^{(1)}, \{u,v\} \subset \eta, \{u\} \in S^2, \{v\} \in S^2} m(\eta) \right)$$

$$= \sum_{\{u,v\} \in X^{(1)}, \{u\} \in S^2, \{v\} \in S^2} \left(\sum_{\sigma \in X^{(4)}, \{u,v\} \subset \sigma} m(\eta) \right)$$

$$= \frac{1}{2} \sum_{\{u,v\} \in X^{(1)}, \{u\} \in S^2, \{v\} \in S^2} m(\tau)$$

$$= \frac{1}{2} m(S^2, S^2).$$

28

This inequality allows us to bound the contribution of simplices in K_2^3, K_3^3, K_4^3 to the sum of the norms of the localizations:

$$\sum_{\{v\}\in S^2} \|d_{\{v\}}\phi_{\{v\}}\| = \sum_{\{v\}\in S^2}\left(\sum_{\tau\in X_{\{v\}},\tau\in supp(d_{\{v\}}\phi_{\{v\}})} m_{\{v\}}(\tau)\right) \leq$$

$$\sum_{\{v\}\in S^2}\left(\sum_{\tau\in X_{\{v\}},\tau\in supp(d_{\{v\}}\phi_{\{v\}}),\{v\}\cup\tau\in K_1^3} m_{\{v\}}(\tau)\right) + 2m(K_2^3) + 3m(K_3^3)$$

$$+ 4m(K_4^3) \leq$$

$$\sum_{\{v\}\in S^2}\left(\sum_{\tau\in X_{\{v\}},\tau\in supp(d_{\{v\}}\phi_{\{v\}}),\{v\}\cup\tau\in K_1^3} m_{\{v\}}(\tau)\right) + m(S^2, S^2) \leq$$

$$\sum_{\{v\}\in S^2}\left(\sum_{\tau\in X_{\{v\}},\tau\in supp(d_{\{v\}}\phi_{\{v\}}),\{v\}\cup\tau\in K_1^3} m_{\{v\}}(\tau)\right) + \frac{\varepsilon_2}{10}\|\phi\| =$$

$$\sum_{\{v\}\in S^2}\left(\sum_{\eta\in K_1^3,\{v\}\subset\eta,(\eta\setminus\{v\})\in supp(d_{\{v\}}\phi_{\{v\}})} m(\eta)\right) + \frac{\varepsilon_2}{10}\|\phi\|.$$

Therefore, the main contribution to $\sum_{\{v\}\in S^2}\|d_{\{v\}}\phi_{\{v\}}\|$ comes from simplices in K_1^3:

$$\sum_{\{v\}\in S^2} \|d_{\{v\}}\phi_{\{v\}}\| - \frac{\varepsilon_2}{10}\|\phi\| \leq \sum_{\{v\}\in S^2}\left(\sum_{\eta\in K_1^3,\{v\}\subset\eta,(\eta\setminus\{v\})\in supp(d_{\{v\}}\phi_{\{v\}})} m(\eta)\right). \tag{7}$$

Next, observe that for every $\eta\in X^{(3)}$ and for every $\{v\}\subset\eta$ we have that

$$d\phi(\eta) = d_{\{v\}}\phi_{\{v\}}(\eta\setminus\{v\}) + \phi(\eta\setminus\{v\}),$$

where the addition above is in $\mathbb{F}_2$. Note that for $\eta\in K_1^3$ and $\{v\}\subset\eta, \{v\}\in S^2$ we have that $\eta\setminus\{v\}\in K_0^2$. Therefore

$$\|d\phi\| - \sum_{\{v\}\in S^2}\left(\sum_{\eta\in K_1^3,\{v\}\subset\eta,(\eta\setminus\{v\})\in supp(d_{\{v\}}\phi_{\{v\}})} m(\eta)\right) \geq$$

$$\sum_{\eta\in K_1^3\cap supp(d\phi)} m(\eta) - \sum_{\{v\}\in S^2}\left(\sum_{\eta\in K_1^3,\{v\}\subset\eta,(\eta\setminus\{v\})\in supp(d_{\{v\}}\phi_{\{v\}})} m(\eta)\right) =$$

$$\sum_{\eta\in K_1^3\cap supp(d\phi)} m(\eta) - \sum_{\eta\in K_1^3}\left(\sum_{\{v\}\in S^2,\{v\}\subset\eta,\eta\setminus\{v\}\in supp(d_{\{v\}}\phi_{\{v\}})} m(\eta)\right) \geq$$

$$- \sum_{\eta\in K_1^3,d\phi(\eta)=0,d_{\{v\}}\phi_{\{v\}}(\eta\setminus\{v\})=1 \text{ for } \{v\}\in S^2,\{v\}\subset\eta} m(\eta) \geq$$

$$-m(K_0^2\cap supp(\phi)) \geq -\alpha m(K_1^2\cap supp(\phi)).$$

Combined with (7), this yields

$$\|d\phi\| \geq \sum_{\{v\}\in S^2}\|d_{\{v\}}\phi_{\{v\}}\| - \frac{\varepsilon_2}{10}\|\phi\| - \alpha m(K_1^2\cap supp(\phi)). \tag{8}$$

Note that

$$\sum_{\{v\}\in S^2} \|\phi_{\{v\}}\| \geq m(K_1^2 \cap supp(\phi)).$$

Combine this with (8) and (6) and get

$$\|d\phi\| \geq \sum_{\{v\}\in S^2} \|d_{\{v\}}\phi_{\{v\}}\| - \frac{\varepsilon_2(1+\alpha)}{9} \sum_{\{v\}\in S^2} \|\phi_{\{v\}}\| - \alpha \sum_{\{v\}\in S^2} \|\phi_{\{v\}}\|.$$

Therefore

$$\|d\phi\| \geq \sum_{\{v\}\in S^2} \|d_{\{v\}}\phi_{\{v\}}\| - \left(\frac{\varepsilon_2(1+\alpha)}{9} + \alpha\right) \sum_{\{v\}\in S^2} \|\phi_{\{v\}}\|.$$

Inequality 2: For $i = 0, 1, 2, 3$ denote

$$L_i = \{\sigma \in X^{(2)} : \sigma \text{ has } i \ \delta - \text{thick edges}\}.$$

Observe that if for $\{u, v, w\} \in X^{(2)}$ the edge $\{u, v\}$ is δ-thick, then the vertex u will be δ-thick in $X_{\{v\}}$ and the vertex v will be δ-thick in $X_{\{u\}}$. By this observation we get that every $\sigma \in L_3$ will have an edge with 2 δ-thick vertices in every link. By lemma 7.4, this implies

$$m(K_1^2 \cap L_3 \cap supp(\phi)) \leq \sum_{v\in R^2} 2\varepsilon_1 \|\phi_{\{v\}}\| \leq 6\varepsilon_1 \|\phi\| = \frac{\varepsilon_2}{10}\|\phi\|.$$

Similarly

$$m(K_0^2 \cap (L_2 \cup L_3) \cap supp(\phi)) \leq \frac{\varepsilon_2}{10}\|\phi\|.$$

Next, we observe that

$$\sum_{\tau \in A_\delta} \|\phi_\tau\| = 3m(L_0 \cap supp(\phi)) + 2m(L_1 \cap supp(\phi)) + m(L_2 \cap supp(\phi)) \geq$$

$$2m(K_0^2 \cap (L_0 \cup L_1) \cap supp(\phi)) + m(K_1^2 \cap (L_0 \cup L_1 \cup L_2) \cap supp(\phi)) \geq$$

$$\left(2m(K_0^2 \cap supp(\phi)) - 2m(K_0^2 \cap (L_2 \cup L_3) \cap supp(\phi))\right)$$
$$+ \left(m(K_1^2 \cap supp(\phi)) - m(K_1^2 \cap L_3 \cap supp(\phi))\right) \geq$$

$$2m(K_0^2 \cap supp(\phi)) + m(K_1^2 \cap supp(\phi))$$
$$- \frac{3\varepsilon_2}{10}\|\phi\| \geq (2\alpha + 1)m(K_1^2 \cap supp(\phi)) - \frac{3\varepsilon_2}{10}\|\phi\|.$$

Note that by (5), we have that

$$(1 - \frac{\varepsilon_2}{10})\|\phi\| \leq (1+\alpha)m(K_1^2 \cap supp(\phi)).$$

Combined with the above inequality, this yields

$$\sum_{\tau \in A_\delta} \|\phi_\tau\| \geq 3\left(\frac{1+2\alpha}{3+3\alpha} - \frac{2\varepsilon_2}{15}\right)\|\phi\|.$$

Therefore ϕ is $(\frac{1+2\alpha}{3+3\alpha} - \frac{2\varepsilon_2}{15}, \delta)$-thin.

Therefore, we can conclude the two inequalities analysed above as follows: for every ϕ with α defined as above, we have that

1. For the set S^2 as above

$$\forall \{v\} \in S^2, \|\phi_{\{v\}}\| \geq C_1' m(X_{\{v\}}^{(1)}),$$

$$\|\phi\| \leq \frac{10(1+\alpha)}{9} \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|,$$

and

$$\|d\phi\| \geq \sum_{\{v\} \in S^2} \|d_{\{v\}} \phi_{\{v\}}\| - (\frac{\varepsilon_2(1+\alpha)}{9} + \alpha) \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|.$$

2. ϕ is $(\frac{1+2\alpha}{3+3\alpha} - \frac{2\varepsilon_2}{15}, \delta)$-thin.

Therefore if $\alpha \leq \varepsilon_2$ (using the fact that $\varepsilon_2 \leq 1$), we have that

$$\|\phi\| \leq \frac{20}{9} \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|.$$

We also have that

$$\|d\phi\| \geq \sum_{\{v\} \in S^2} \|d_{\{v\}} \phi_{\{v\}}\| - \frac{11\varepsilon_2}{9} \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|.$$

and if $\alpha \geq \varepsilon_2$ (using the fact that $\varepsilon_2 \leq 1$), we have that ϕ is $(\frac{1}{3} + \frac{\varepsilon_2}{15}, \delta)$-thin. Last, we note that in the case where $m(K_1^2 \cap supp(\phi)) = 0$, repeating the above calculation of the second inequality shows that

$$\sum_{\tau \in A_\delta} \|\phi_\tau\| \geq 3 \left(\frac{2}{3} - \frac{2\varepsilon_2}{15} \right) \|\phi\|,$$

and therefore in this case ϕ is $(\frac{2}{3} - \frac{2\varepsilon_2}{15}, \delta)$-thin (note that $\varepsilon_2 \leq 1$ and therefore the theorem holds for this case as well). $\qquad \square$

Next, we'll add a further assumption about the coboundary expansion of the links of vertices to deduce isoperimetric inequalities in the case $k = 2$:

Theorem 7.9. *For every $\epsilon > 0$, there are constants $\theta_2 = \theta_2(\epsilon) < 1$, $C_2 = C_2(\epsilon) < 1$, $\varepsilon(\epsilon) > 0$, such that for every simplicial complex X, if*

$$\min\{\lambda_0, \lambda_1, \lambda_2\} \geq \theta_2,$$

and

$$\forall \{v\} \in X^{(0)}, \epsilon_1(X_{\{v\}}) \geq \epsilon,$$

where $\epsilon_1(X_{\{v\}})$ is the 1-coboundry expansion of $X_{\{v\}}$. Then for every $0 \neq \phi \in C^2(X, \mathbb{F}_2)$, we have that

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \leq C_2 m(X^{(2)}) \right) \Rightarrow \|d\phi\| \geq \frac{3\epsilon}{10} \|\phi\| > 0.$$

Proof. Choose $\varepsilon_2 = \frac{35\epsilon}{100}, \delta = \frac{\epsilon}{1000}$ and let $C_1' = C_1'(\varepsilon_2, \delta), C_2' = C_2'(\varepsilon_2, \delta), \theta_2' = \theta_2'(\varepsilon_2, \delta)$ be as in theorem 7.8 above. Choose

$$\varepsilon = \min\left\{\frac{C_1'}{4}, \frac{\epsilon}{1000}\right\},$$

$$\theta_2 = \max\left\{\theta_2', \frac{2 + \frac{3\epsilon}{10}}{2 + \frac{61\epsilon}{2000}}\right\},$$

$$C_2 = C_2'.$$

Fix $0 \neq \phi \in C^2(X, \mathbb{F}_2)$ such that ϕ is ε-locally minimal and $\|\phi\| \leq C_2 m(X^{(2)})$. By theorem 7.8 above at least one the the following occurs:

1. There is a set $S^2 \subset X^{(0)}$ such that

$$\forall \{v\} \in S^2, \|\phi_{\{v\}}\| \geq C_1' m(X^{(1)}_{\{v\}}),$$

$$\sum_{\{v\} \in S^2} \|\phi_{\{v\}}\| \geq \frac{9}{20}\|\phi\|, \tag{9}$$

and

$$\|d\phi\| \geq \sum_{\{v\} \in S^2} \|d_{\{v\}}\phi_{\{v\}}\| - \frac{11\varepsilon_2}{9}\sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|. \tag{10}$$

2. ϕ is $(\frac{1}{3} + \frac{\varepsilon_2}{15}, \delta)$-thin.

We'll prove the needed inequality for each case.
Case 1: If there is a set S^2 as mentioned above, we have for each $\{v\} \in S^2$, we have that

$$\|\phi_{\{v\}}\| \geq C_1' m(X^{(1)}_{\{v\}}) = \frac{C_1'}{2}m(X^{(0)}_{\{v\}}) = \frac{C_1'}{2}m(\{v\}).$$

By the fact that ϕ is ε-locally minimal and that $\varepsilon \leq \frac{C_1'}{4}$, we have for every $\{v\} \in S^2$ that

$$\min\{\|\phi_{\{v\}} - \varphi\| : \varphi \in B^1(X_{\{v\}}, \mathbb{F}_2)\} \geq$$
$$\|\phi_{\{v\}}\| - \frac{C_1'}{4}m(\{v\}) \geq \|\phi_{\{v\}}\| - \frac{1}{2}\|\phi_{\{v\}}\| = \frac{1}{2}\|\phi_{\{v\}}\|.$$

By our assumption on $\epsilon_1(X_{\{v\}})$, we therefore have for each $\{v\} \in S^2$ that

$$\|d_{\{v\}}\phi_{\{v\}}\| \geq \epsilon \min\{\|\phi_{\{v\}} - \varphi\| : \varphi \in B^1(X_{\{v\}}, \mathbb{F}_2)\} \geq \frac{\epsilon}{2}\|\phi_{\{v\}}\|.$$

Next, combine the above inequality (10) to get

$$\|d\phi\| \geq \sum_{\{v\} \in S^2} \|d_{\{v\}}\phi_{\{v\}}\| - \frac{11\varepsilon_2}{9}\sum_{\{v\} \in S^2} \|\phi_{\{v\}}\| \geq \sum_{\{v\} \in S^2} \|\phi_{\{v\}}\|\left(\frac{\epsilon}{2} - \frac{11\varepsilon_2}{9}\right).$$

We choose $\varepsilon_2 = \frac{35\epsilon}{100}$ and therefore, this yields

$$\|d\phi\| \geq \frac{65\epsilon}{900}\sum_{\{v\} \in S^2} \|\|\phi_{\{v\}}\|.$$

By (9), we get that

$$\|d\phi\| \geq \frac{65\epsilon}{900} \frac{9}{20} \|\phi\| = \frac{65\epsilon}{2000} \|\phi\| > \frac{3\epsilon}{10} \|\phi\|,$$

as needed.

Case 2: If ϕ is $(\frac{1}{3} + \frac{\varepsilon_2}{15}, \delta)$-thin, then by lemma 7.2, we have that

$$\|d\phi\| \geq \left(\left(\frac{\frac{1}{3} + \frac{\varepsilon_2}{15} + 1}{2} - \delta - \frac{\varepsilon}{2} \right) 3\lambda_2 - 2 \right) \|\phi\|.$$

Which can be simplified to

$$\|d\phi\| \geq \left(\left(2 + \frac{\varepsilon_2}{10} - 3\delta - \frac{3\varepsilon}{2} \right) \lambda_2 - 2 \right) \|\phi\|.$$

We chose $\varepsilon_2 = \frac{35\epsilon}{100}, \delta = \frac{\epsilon}{1000}, \varepsilon \leq \frac{\epsilon}{1000}$ and therefore this reads

$$\|d\phi\| \geq \left(\left(2 + \frac{35\epsilon}{1000} - \frac{3\epsilon}{1000} - \frac{3\epsilon}{2000} \right) \lambda_2 - 2 \right) \|\phi\|.$$

After more simplifying we get

$$\|d\phi\| \geq \left(\left(2 + \frac{61\epsilon}{2000} \right) \lambda_2 - 2 \right) \|\phi\|.$$

We chose $\theta \geq \frac{2 + \frac{3\epsilon}{10}}{2 + \frac{61\epsilon}{2000}}$ and therefore we have that

$$\|d\phi\| \geq \left(2 + \frac{3\epsilon}{10} - 2 \right) \|\phi\| = \frac{3\epsilon}{10} \|\phi\|,$$

as needed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Combining the corollary 7.3, theorem 7.5 with theorem 2.10, yields the following (which proves theorem 1.4 stated in the introduction):

Theorem 7.10. *For every $n > 1$, there are constants $\frac{n-1}{n} < \Lambda_1 < 1, \varepsilon > 0, C_1 > 0$ such that for every weighted simplicial complex X of dimension $n > 1$ with λ-local spectral expansion, if $\lambda \geq \Lambda_1$, we have:*

1. *For every $0 \neq \phi \in C^0(X, \mathbb{F}_2)$*

$$(\phi \text{ is } \varepsilon\text{-locally minimal}) \Rightarrow \|d\phi\| > 0.$$

2. *For every $0 \neq \phi \in C^1(X, \mathbb{F}_2)$*

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \leq C_1 m(X^{(1)}) \right) \Rightarrow \|d\phi\| \geq \frac{1}{4} \|\phi\| > 0.$$

Combing the above with theorem 7.9 (and adding further assumptions regarding the coboundary expansion of the links of vertices) yields the following:

Theorem 7.11. *For every $n > 2$, $\epsilon > 0$, there are constants $\frac{n-1}{n} < \Lambda_2 < 1, \varepsilon > 0, C_1 > 0, C_2 > 0$ such that for every weighted simplicial complex X of dimension $n > 2$ with λ-local spectral expansion, if $\lambda \geq \Lambda_2$ and if*

$$\forall \{v\} \in X^{(0)}, \epsilon_1(X_{\{v\}}) \geq \epsilon,$$

(where $\epsilon_1(X_{\{v\}})$ is the 1-coboundry expansion of $X_{\{v\}}$), we have that

1. *For every $0 \neq \phi \in C^0(X, \mathbb{F}_2)$*

$$(\phi \text{ is } \varepsilon\text{-locally minimal}) \Rightarrow \|d\phi\| > 0.$$

2. *For every $0 \neq \phi \in C^1(X, \mathbb{F}_2)$*

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \leq C_1 m(X^{(1)}) \right) \Rightarrow \|d\phi\| \geq \frac{1}{4}\|\phi\| > 0.$$

3. *For every $0 \neq \phi \in C^2(X, \mathbb{F}_2)$*

$$\left(\phi \text{ is } \varepsilon\text{-locally minimal and } \|\phi\| \leq C_2 m(X^{(2)}) \right) \Rightarrow \|d\phi\| \geq \frac{3\epsilon}{10}\|\phi\| > 0.$$

Combining the above theorem with lemma 6.4, gives the following theorem:

Theorem 7.12. *For $n > 2$, $\epsilon > 0$ there are constants $\frac{n-1}{n} < \Lambda_2 = \Lambda_2(n) < 1$, $1 \leq M, c = c(\Lambda_2, M, \epsilon)$, such that for any n-dimensional complex X, if:*

1. *X has λ-local spectral expansion and $\lambda \geq \Lambda_2$.*

2.
$$\forall \{v\} \in X^{(0)}, \epsilon_1(X_{\{v\}}) \geq \epsilon.$$

3.
$$\frac{\sup_{\sigma \in X^{(2)}} |\{\eta \in X^{(n)} : \sigma \subset \eta\}|}{\inf_{\sigma \in X^{(2)}} |\{\eta \in X^{(n)} : \sigma \subset \eta\}|} \leq M.$$

Then the 2-skeleton of X has c-topological overlapping.

8 Topological overlapping for 2-skeletons of affine buildings

The main difficulty of applying theorem 7.12 in order to construct a sequence of 2-dimensional topological expanders is the second condition stated in the theorem, i.e., the condition bounding ϵ_1 for each link of each vertex. Fortunately, such bounds exist for affine buildings that arise from BN-pairs. The subject of buildings is far to wide to present in the context of this article and the interested reader is referred to [AB08] and references therein. Here we shall assume knowledge of the basic facts of BN-pairs and affine buildings.
The main result we'll use is the bound on the coboundary expansion of spherical buildings that arise from BN-pairs. This result was already mentioned in [Gro10][page 457] and an explicit proof can be found in [LMM14][Section 3]:

Theorem 8.1. *[LMM14][Corollary 3.6] Let G be a group with a BN-pair and a spherical (i.e., finite) (W, S) Coxeter group with $|S| = n + 1$. Let Y be the (finite) n-dimensional spherical building that arises for the BN-pair. Then for every $0 \leq k \leq n - 1$, we have that*

$$\epsilon_k(Y) \geq \left(\binom{n+1}{k+2}^2 |W| \right)^{-1}.$$

We note that the bound on ϵ_1 depends only on the type of the building and not on the thickness of the building. Next, let $\widetilde{X}$ be an affine n-dimensional building that arises from a group G with an affine BN-pair constructed over a non-archimedean local field F, i.e., F is a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$. We recall that for every type of affine, we can choose p or q (related to F) to be large as we want. This choice determines the thickness of the building, but does not effect the type of the building (W stays the same). Note that all the links of $\widetilde{X}$ (excluding $\widetilde{X}$ itself) are spherical and therefore for X we have the desired bound on ϵ_1 on the links of all vertices (as noted - this bound is not affected by the choice of the field F).

Next, we recall that the spectral gaps of all 1-dimensional spherical buildings were computed explicitly in [FH64] and for every type of spherical building with thickness t, the spectral gap is $\geq 1 - O(t^{-\frac{1}{2}})$. Therefore choosing $F = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$ with p or q large enough ensures that the spectral gap of all 1 dimensional links of X are greater that Λ_2 for theorem 7.12. Therefore, by taking quotients of $\widetilde{X}$ that do not change the links of $\widetilde{X}$ we can get a topological expander:

Corollary 8.2. *Let $\widetilde{X}$ be an n dimensional $(n > 2)$ affine building that arises from group G with an affine BN-pair constructed over a non-archimedean local field $F = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$ with p or q large. Let W be the Weyl group of the BN-pair. Let Γ be a discrete subgroup of G acting cocompactly on X such that*

$$\forall \{v\} \in \widetilde{X}^{(0)}, \forall g \in \Gamma, d(g.v, v) > 2,$$

where d is the distance with respect to the metric of the 1-skeleton on $\widetilde{X}$. Then there is $c = c(p \text{ or } q, W) > 0$ such that $X = \widetilde{X}/\Gamma$ has c-topological overlapping.

Proof. We apply theorem 7.12. Note that the condition

$$\forall \{v\} \in X^{(0)}, \forall g \in \Gamma, d(g.v, v) > 2,$$

ensures that all $1, ..., n-1$-dimensional links of $\widetilde{X}$ are isomorphic to links of X. Therefore choosing p (or q) large enough ensures that X/Γ has λ-local spectral gap, where $\lambda \geq \Lambda_2$. Also, it ensures that for every vertex v of X we have that

$$\epsilon_1(X_{\{v\}}) \geq \left(\binom{n+1}{3}^2 |W| \right)^{-1}.$$

Last, the constant M in the theorem can be computed by the type of the building as a function of p or q (if $\widetilde{X}$ is of type $\widetilde{A}_n$, then $M = 1$, but otherwise is it a function of p or q). $\square$

A Cheeger inequality for weighted graphs

The aim of this appendix is to review the basic definition of a weighted graph and to state and prove the Cheeger inequality (and some of its consequences for weighted graphs). This appendix doesn't contain any new results and we provide the proofs merely for the sake of completeness.

For a graph $G = (V, E)$, a weight function is a function $m : V \cup E \to \mathbb{R}^+$ such that

$$\forall v \in V, m(v) = \sum_{e \in E, v \in e} m(e).$$

Denote the ordered edges of the graph as $\Sigma(1)$, i.e.,

$$\Sigma(1) = \{(v, u) : v, u \in V, \{u, v\} \in E\}.$$

Define

$$C^0(G, \mathbb{R}) = \{\phi : V \to \mathbb{R}\},$$

$$C^1(G, \mathbb{R}) = \{\phi : \Sigma(1) \to \mathbb{R} : \forall (u, v) \in \Sigma(1), \phi((u, v)) = -\phi((v, u))\}.$$

Define the differential $d : C^0(G, \mathbb{R}) \to C^1(G, \mathbb{R})$ as

$$d\phi((u, v)) = \phi(u) - \phi(v).$$

Define inner products on $C^0(G, \mathbb{R}), C^1(G, \mathbb{R})$ as

$$\forall \phi, \psi \in C^0(G, \mathbb{R}), \langle \phi, \psi \rangle = \sum_{v \in V} m(v)\phi(v)\psi(v),$$

$$\forall \phi, \psi \in C^1(G, \mathbb{R}), \langle \phi, \psi \rangle = \sum_{(u,v) \in \Sigma(1)} \frac{1}{2}m(\{u, v\})\phi((u, v))\psi((u, v)).$$

Denote $\|.\|$ to be the norm with respect to the inner products defined above (this should not be confused with our use of $\|.\|$ in the body of this paper). Then $C^0(G, \mathbb{R}), C^1(G, \mathbb{R})$ are Hilbert spaces and we can define $d^* : C^1(G, \mathbb{R}) \to C^0(G, \mathbb{R})$ as the adjoint operator to d. The graph Laplacian is defined as

$$\Delta^+ : C^0(G, \mathbb{R}) \to C^0(G, \mathbb{R}), \Delta^+ = d^*d.$$

Note that by its definition, Δ^+ is a positive operator.

Proposition A.1. *Let $G = (V, E)$ be a graph with a weight function m. Then:*

1. *$d^* : C^1(G, \mathbb{R}) \to C^0(G, \mathbb{R})$ can be written explicitly as*

$$\forall \psi \in C^1(G, \mathbb{R}), \forall v \in V, d^*\phi(v) = \sum_{u \in V, (v,u) \in \sigma(1)} \frac{m(\{u, v\})}{m(v)}\phi((v, u)).$$

2. *Δ^+ can be written explicitly as*

$$\forall \phi \in C^0(G, \mathbb{R}), \forall v \in V, \Delta^+\phi(v) = \phi(v) - \frac{1}{m(v)} \sum_{u \in V, \{u,v\} \in E} m(\{u, v\})\phi(u).$$

3. *If G is connected, then $\Delta^+\phi = 0$ if and only if ϕ is constant.*

Proof.   1. Let $\phi \in C^0(G, \mathbb{R})$ and $\psi \in C^1$, then

$$
\begin{aligned}
\langle d\phi, \psi \rangle &= \sum_{(u,v)\in\Sigma(1)} \frac{1}{2} m(\{u,v\}) d\phi((u,v)) \psi((u,v)) \\
&= \sum_{(u,v)\in\Sigma(1)} \frac{1}{2} m(\{u,v\})(\phi(u) - \phi(v)) \psi((u,v)) \\
&= \sum_{(u,v)\in\Sigma(1)} \frac{1}{2} m(\{u,v\}) \left(\phi(u)\psi((u,v)) + \phi(v)\psi((v,u)) \right) \\
&= \sum_{v\in V} m(v)\phi(v) \sum_{(v,u)\in\Sigma(1)} \frac{m(\{u,v\})}{m(v)} \psi((v,u)) \\
&= \langle \phi, d^*\psi \rangle.
\end{aligned}
$$

2. Let $\phi \in C^0(G, \mathbb{R})$ and $v \in V$, then

$$
\begin{aligned}
\Delta^+ \phi(v) &= d^* d\phi(v) \\
&= \sum_{u\in V, (v,u)\in\sigma(1)} \frac{m(\{u,v\})}{m(v)} d\phi((v,u)) \\
&= \sum_{u\in V, (v,u)\in\sigma(1)} \frac{m(\{u,v\})}{m(v)} (\phi(v) - \phi(u)) \\
&= \phi(v) - \sum_{u\in V, (v,u)\in\sigma(1)} \frac{m(\{u,v\})}{m(v)} \phi(u) \\
&= \phi(v) - \frac{1}{m(v)} \sum_{u\in V, \{u,v\}\in E} m(\{u,v\})\phi(u).
\end{aligned}
$$

3. Let $\mathbb{1} \in C^0(G, \mathbb{R})$, the constant function $\mathbb{1}(v) = 1, \forall v \in V$. Then $d\mathbb{1} \equiv 0$ and therefore $\Delta^+ \mathbb{1} = 0$. On the other hand, if $\phi \in C^0(G, \mathbb{R})$ is a function such that $\Delta^+ \phi = 0$, we can apply the maximum principle: let $v_0 \in V$ such that

$$
\forall v \in V, \phi(v_0) \geq \phi(v).
$$

Next,

$$
0 = \Delta^+ \phi(v_0) = \phi(v_0) - \frac{1}{m(v_0)} \sum_{u\in V, \{u,v_0\}\in E} m(\{u,v_0\})\phi(u),
$$

implies that

$$
\phi(v_0) = \frac{1}{m(v_0)} \sum_{u\in V, \{u,v_0\}\in E} m(\{u,v_0\})\phi(u).
$$

The right hand side of the equation is an average $\phi(u)$'s such that for every u, $\phi(v_0) \geq \phi(u)$. Therefore the equality implies that

$$
\forall u \in V, \{u, v_0\} \in E \Rightarrow \phi(u) = \phi(v_0).
$$

Iterating this argument (using the fact that G is connected) yields that ϕ must be constant.

$\square$

As a result of the above proposition, if G is connected, then Δ^+ has the eigenvalue 0 with multiplicity 1 and all the other eigenvalues are strictly positive. Denote by $\lambda(G)$ the smallest positive eigenvalue. Next, we can state the Cheeger inequality:

Proposition A.2. *Let $G = (V, E)$ be a connected graph. We introduce the following notations: For $\emptyset \neq U \subseteq V$ denote*

$$m(U) = \sum_{v \in U} m(v).$$

For $\emptyset \neq U_1, U_2 \subseteq V$ denote

$$m(U_1, U_2) = \sum_{u_1 \in U_1, u_2 \in U_2, \{u_1, u_2\} \in E} m(\{u_1, u_2\}).$$

Then:

1. *(Cheeger inequality) For every $\emptyset \neq U \subsetneq V$, we have that*

$$m(U, V \setminus U) \geq \lambda(G) \frac{m(U) m(V \setminus U)}{m(V)}.$$

2. *For every $\emptyset \neq U \subsetneq V$, we have that*

$$\frac{m(U)}{2} \left(1 - \lambda(G) \frac{m(V \setminus U)}{m(V)} \right) \geq m(U, U).$$

Proof.   1. Let $\emptyset \neq U \subsetneq V$. Define $\phi \in C^0(G, \mathbb{R})$ as

$$\phi(u) = \begin{cases} \dfrac{1}{m(U)} & u \in U \\ \dfrac{-1}{m(V \setminus U)} & u \in V \setminus U \end{cases}.$$

Recall that $\mathbb{1}$ is the constant function 1 and note that

$$\begin{aligned}
&\langle \phi, \mathbb{1} \rangle \\
&= \sum_{u \in V} m(u) \phi(u) 1 \\
&= \sum_{u \in U} m(u) \frac{1}{w(U)} + \sum_{u \in V \setminus U} m(u) \frac{-1}{m(V \setminus U)} \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

Therefore $\phi \perp \mathbb{1}$. Recall that Δ^+ is a positive operator (in particular, it has orthogonal eigenfunction), therefore

$$\langle \Delta^+ \phi, \phi \rangle \geq \lambda(G) \|\phi\|^2.$$

Next, note the following

$$\langle \Delta^+\phi, \phi \rangle = \langle d\phi, d\phi \rangle$$

$$= \sum_{(u,v)\in\Sigma(1), u\in U, v\in V\setminus U} \frac{1}{2}m(\{u,v\})\left(\frac{1}{m(U)} + \frac{1}{m(V\setminus U)}\right)^2$$

$$= m(U, V\setminus U)\left(\frac{1}{m(U)} + \frac{1}{m(V\setminus U)}\right)^2.$$

Also,

$$\|\phi\|^2 = \sum_{u\in U} m(u)\frac{1}{m(U)^2} + \sum_{u\in V\setminus U} m(u)\frac{1}{m(V\setminus U)^2} = \frac{1}{m(U)} + \frac{1}{m(V\setminus U)}.$$

Therefore, the inequality

$$\langle \Delta^+\phi, \phi \rangle \geq \lambda(G)\|\phi\|^2,$$

yields

$$m(U, V\setminus U)\left(\frac{1}{m(U)} + \frac{1}{m(V\setminus U)}\right)^2 \geq \left(\lambda(G)\frac{1}{m(U)} + \frac{1}{m(V\setminus U)}\right),$$

which in turn yields

$$m(U, V\setminus U) \geq \lambda(G)\frac{1}{\frac{1}{m(U)} + \frac{1}{m(V\setminus U)}}$$

$$= \lambda(G)\frac{m(U)m(V\setminus U)}{m(U) + m(V\setminus U)}$$

$$= \lambda(G)\frac{m(U)m(V\setminus U)}{m(V)},$$

and we are done.

2. Let $\emptyset \neq U \subsetneq V$. Note that for every $u \in U$ we have that

$$m(u) = \sum_{v\in V, \{u,v\}\in E} m(\{u,v\}) = \sum_{v\in U, \{u,v\}\in E} m(\{u,v\}) + \sum_{v\in V\setminus U, \{u,v\}\in E} m(\{u,v\}).$$

Summing on all $u \in U$ we get that

$$m(U) = \sum_{u\in U}\sum_{v\in U, \{u,v\}\in E} m(\{u,v\}) + \sum_{u\in U}\sum_{v\in V\setminus U, \{u,v\}\in E} m(\{u,v\})$$

$$= 2m(U,U) + m(U, V\setminus U).$$

Using the Cheeger inequality proven above, we get that

$$m(U) \geq 2m(U,U) + \lambda(G)\frac{m(U)m(V\setminus U)}{m(V)}.$$

This yields that

$$\frac{m(U)}{2}\left(1 - \lambda(G)\frac{m(V\setminus U)}{m(V)}\right) \geq m(U,U).$$

$\square$

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